The first law of mechanics for spinning binary systems

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Abstract

The first law of general relativistic mechanics for astrophysical binary systems is a powerful tool for waveform construction in the ongoing theoretical work in Gravitational Wave research. It links the global quantities of the binary system with the local parameters of its constituents, by expressing the relations of their variations. The present work aims to derive a first law for binary systems of spinning point particles endowed with internal structure in circular orbit, within certain approximation schemes in General Relativity separating the conservative dynamics from the dissipative one and achieving asymptotic flatness. In particular, the spacetime metric is taken to be the one sourced by the stress energy momentum (SEM) tensor of the binary particles at quadrupole order in multipole expansion, in a regularized sense. First, a description of spinning point particles in curved spacetime is reviewed, via a general Lagrangian approach and Tulczyjew's reduction formalism for gravitational skeletons. A sketch of generalization to quadrupolar particles by the latter approach is presented. The skeleton formalism in turn is intended to be used to prove that the particles' worldlines coincide with the integral lines of the helical Killing vector field by which the circular orbit is mathematically described, and that the particles' reduced multipole moments are Lie-dragged along their worldlines. The general approach for deriving the first law using differential forms is presented, for which the classic result of first law for a stationary axisymmetric black hole is re-derived, to demonstrate the method's utility. Such a strategy is further applied to the aforementioned skeleton SEM tensor for the point particle binary system, for which a first law to dipolar order established before is re-derived. Preliminary results for its extension to quadrupole order are presented, as well as envisaged mathematical and physical strategies to proceed. Such a result, once obtained, should have implications on studying tidal deformations of binary neutron stars.

Contents

1 Introduction

Since the first direct detection of Gravitational Wave (GW) generated by the merger of binary black holes (BHs) in 2015[1], the field of General Relativity (GR) and cosmology has witnessed an outburst of activity thanks to the opening of GW astrophysics and GW cosmology. The first joint observation of electromagnetic signal and GW by the event GW170817 for binary neutron star (NS) initiated the study of multi-messenger astronomy[2]. Many questions about astrophysics, nuclear physics and fundamental physics have been studied thanks to this detection, notably the tidal deformation of NS[3] and the equation of state of extremely dense matter[4], a tight selection of alternative theories of gravity[5, 6, 7]. With more and more events detected by the LIGO-Virgo-KAGRA collaboration through the 4 observation runs by now and in the future[8, 9], along with the planned space-based GW mission like LISA[10, 11, 12], many scientific questions are being vigorously investigated thanks to this new powerful tool, such as further test of GR in the strong field regime[13], UV completion of GR and constraints on its effective field theory extension[14], an independent measure of Hubble parameter[15], and the relic GWs from phase transitions in the early universe[16], to name just a few.

For continuously exploring the aforementioned scientific questions, accurate waveforms from theoretical modelling of the sources of GWs especially for astrophysical binary systems are of pivotal importance. Faithful waveform models are needed for the matched filtering technique in data analysis to extract the signal from a myriad of noises[17]. In addition, the accuracy of waveforms is also crucial for properly inferring the parameters of GW sources in order to verify or check the validities of theories. Many approaches for waveform modelling have been developed by the theory community. The post-Newtonian (PN) method is an approximation scheme for weakly self-gravitating and slowly moving sources. It performs a first principle GR calculation based on an expansion of (v/c), with v being the typical velocity of the source internal motion and c the speed of light[18]. The effective one-body (EOB) method developed by Buonanno and Damour maps the metric in the geometry of binary inspirals into an effective metric of a single body, while introducing effective potential functions in the one-body metric in a deformed Schwarzschild form[19]. For the merger phase, numerical relativity (NR) is needed to accurately capture the waveforms, by directly solving Einstein field equation numerically[20, 21, 22]. In the case of extreme mass ratio inspiral (EMRI), where the mass of one of the bodies is much smaller than that of its companion, the gravitational self force (GSF) formalism turns out to be an effective description[23]. It relies on a perturbative treatment of corrections brought by the curving of spacetime from the small bodies and performs a calculation in expansion of the mass ratio $q = m_1/m_2 \ll 1$. For all the theoretical tools mentioned above, the so-called first law of mechanics of binary system, which relates the variations of global quantities of the binary and those of its local parameters, can serve as bridges between the various theoretical techniques to facilitate the calculations of one with results from the others.

The first law of BH mechanics was first pioneered by Bardeen, Carter and Hawking (BCH), initially endowed with a thermodynamic interpretation, as a relation between the variations of different quantities for a stationary axisymmetric solution to Einstein field equation[24]. It was later generalized by Wald and Iyer to a general theory of gravity with diffeomorphism invariance, by assigning local geometrical prescription to the BH entropy as the Noether charge corresponding to the horizon Killing field[25, 26]. With the start of construction of kilometer-scale interferometers and the foreseeing future of GW detection in the 1990s[27], a generalization of the first law to binary system began to capture the interest of the theory community. The first generalization to binary system consisting of vacuum or perfect fluid solutions in GR was derived by Friedman, Uryu and Shibata[28]. The variation of the Noether charge associated with the helical Killing field is related to variation of baryon mass, entropy, vorticity for the fluid bodies and of horizon surface area for the BH. In addition to extended bodies, a first law for point particle binary system was derived from a PN framework by Le Tiec, Blanchet and Whiting[29]. The result was subsequently generalized to the schemes of spinning particles[30], Kerr BH with a perturbing moon[31], and eccentric orbit[32]. A key result of the first law for point particle binary is the relation between the global quantities of the system, such as binding energy and total angular momentum, and local parameters of the individual bodies, e.g., redshift parameter introduced by Detweiler[33], the masses and spins of the particles. Such a relation turns out to be helpful in a variety of applications, including the calculations of higher order PN coefficients[29], informing GSF perturbative calculation of global quantities[34]. Corrections to geodesics of test particles in Schwarzschild background caused by the particle's self-field have been computed with the aid of the first law, confirming the frequency shift of the innermost stable circular orbit (ISCO) and innermost bound circular orbirt (IBCO) previously found by numerical computation[34, 35]. The first law has also been used as consistency checks and benchmarks for GSF calculation, especially to its second order[36]. Moreover, the first law informed GSF results can be used to calibrate the EOB potentials[37, 38] as well.

Among the various formalisms to derive the first law for point particle binaries, the use of gravitational skeletonization and the differential geometry based calculations initiated by Ramond and Le Tiec in [39], prove to be relatively straightforward for reaching the first laws, without solving explicitly for the metric from Einstein field equation and dealing with self-field and finite-size regularization. The skeleton formalism is motivated from the multipole expansion for gravitational potential in Newtonian gravity, and is developed mainly by Mathisson[40], Papapetrou[41, 42], Tulczyjew[43] and Dixon[44, 45]. It amounts to modelling the point particles endowed with additional degrees of freedom, such as spins encoded in dipole moments, spin-induced quadrupole and internal structures in the quadrupole moments. The authors of [39] proved, in the context of two quadrupolar particles in circular orbit, the result that the point particles follow helical Killing trajectories previously appearing in [46, 47, 48], but in a more general setting by using Tulczyjew's skeletonization. In addition, it was also shown that the linear momenta, spin and quadrupole tensors of the two particles are Lie-dragged along their worldlines. A first law for spinning particles at dipolar order was derived in their subsequent paper [49], confirming the previous results in [30] using the canonical Arnowitt-Deser-Misner (ADM) formalism. The aim of the present work is to generalize the first law derived in [49] to quadrupole order in the skeleton formalism. Such a picture accounts for modelling the internal structures endowed to the point particles. The first law of mechanics for such a scheme should have implications on modelling binary NS and further study of their tidal deformability and equation of state.

The present work is organized as follows: section 2 reviews the derivations of the equations of motion for multipolar particles. In particular, the Lagrangian approach and Tulczyjew's skeletonization for dipolar particles are presented. A sketch of generalization to quadrupolar particles is mentioned. Section 3 sets up the physical picture of the spinning point particle binary under discussion. Physical arguments and interpretations for the helical Killing trajectories of the particles and the Lie-dragging of their Tulczyjew-reduced multipole moments are stated, while technical proofs are relagated to appendix B. Section 4 reviews the method of Wald *et al* about differential forms for the study of Noether charge associated with a vector generating a diffeomorphism in a diffeomorphism covariant theory for gravity involving matter fields. In particular it will also show the agreement with the first law found by BCH in [24] when the theory is specified to GR. A general variational formula for the case of GR containing a compact matter source described by the SEM tensor T^{ab} is derived. In section 5, the general formula is applied to dipolar particle binary to derive the first law in the picture set up in section 3. Preliminary results and proceeding strategies for the quadrupole first law are presented. Early Latin indices like $a, b, c\dots$ denote abstract tensor indices. Greek indices $\mu, \nu\dots$ are used for tensor component ranging from 0 to 3. Middle Lattin indices $i, j, k\dots$ are used for spatial tensor components. ∇_a denotes the metric compatible derivative operator. Boldface letters without indices denote differential forms, whose rank should be clear or specified from the context. The sans serif index $\mathsf{i} = 1, 2$ is used to label the particles in the binary system. In section 4 and 5, the letter ϕ is used both for dynamical fields and azimuthal angle at spatial infinity. It has become standard in the literature but the context should disthinguish them. The metric signature is taken to be $(-, +, +, +)$. Geometric unit $G = c = 1$ is used throughout the text.

2 Spinning point particles in curved spacetime

This section reviews two approaches for the description of spinning point particles in curved spacetime, the general Lagrangian approach and Tulczyjew's reduction. A generalization to spinning particles with quadrupolar internal structure will be sketched for the latter approach. These formalisms will give the equations of motion for the particles as well as their stress energy momentum (SEM) tensor, which will be used to derive specific versions of the first law later.

2.1 Lagrangian dynamics

The Lagrangian derivation of equation of motion for spinning point particles was achieved by various authors, notably in [50], motivated from and generalizing the results of the case in special relativity[51]. Such a derivation will be reviewed below in more detail than was originally presented.

The degrees of freedom for the spinning particle include its position $x^\mu(\tau)$ on the worldline parametrized by its proper time τ, or equivalently its 4-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, and an orthonormal tetrad ϵ^μ_A ($A, B \dots = 0, 1, 2, 3$ is a label for the tetrad), which in curved spacetime satisfies

$$\epsilon^\mu_A \epsilon_{\mu B} = \eta_{AB}, \qquad \text{or} \qquad \epsilon^A_\mu \epsilon_{\nu A} = g_{\mu\nu}, \tag{2.1}$$

where $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1)$. The action, in the most generic setting, is of the form

$$S[x^\mu, \epsilon^\mu_A] = \int L(u^\mu, \epsilon^\mu_A, \dot{\epsilon}^\mu_A)\, \mathrm{d}\tau. \tag{2.2}$$

Here the over-dot notation is reserved for $\dot{\epsilon}^\mu_A = \mathrm{D}\epsilon^\mu_A/\mathrm{d}\tau \equiv u^\nu \nabla_\nu \epsilon^\mu_A$. The particle is taken as test particle moving in a given spacetime with a metric $g_{\mu\nu}$ in the chart under analysis. Alternatively, the functional dependence of the Lagrangian can also be on the 4-velocity, the spin tensor and the metric[30]. The variation of the latter will give the SEM tensor. This subsection sticks to (2.2), since it's more straightforward to derive the equation of motion. The conjugate momentum and the spin angular momentum are defined from the Lagrangian as

$$p_\mu \equiv \frac{\partial L}{\partial u^\mu} \qquad S^{\mu\nu} \equiv 2\epsilon^{[\mu}_A \frac{\partial L}{\partial \dot{\epsilon}_{\nu]A}}. \tag{2.3}$$

The equation of motion for the spin is most directly obtained from the variational equation $\delta L/\delta\epsilon^\mu_A = 0$, where

$$\frac{\delta L}{\delta \Phi} \equiv \frac{\partial L}{\partial \Phi} - \frac{\mathrm{D}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{\Phi}}, \tag{2.4}$$

with Φ being an arbitrary tensor field of any rank. But because the tetrad has to satisfy the othonormal constraint (2.1), one has to vary with Lagrangian multipliers accounting for the extremizing solution obeying these constraints. Consider the modified Lagrangian

$$\tilde{L} = L + \sum_{0\leqslant\mu\leqslant\nu\leqslant 3} \xi^{\mu\nu}\left(\epsilon^A_\mu \epsilon_{\nu A} - g_{\mu\nu}\right). \tag{2.5}$$

The summation $\sum_{0\leqslant\mu\leqslant\nu\leqslant 3}$ is for only summing over the independent constraints, for which there are 10. From the original 16 degrees of freedom of ϵ^μ_A, only 6 remains, which is the correct number of degrees of freedom for the anti-symmetric $S^{\mu\nu}$. From

$$\frac{\delta \tilde{L}}{\delta \epsilon^\mu_A} = \frac{\partial L}{\partial \epsilon^\mu_A} - \frac{\mathrm{D}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{\epsilon}^\mu_A} + \xi_{\mu\nu}\epsilon^{\nu A} = 0 \tag{2.6a}$$

$$\frac{\delta \tilde{L}}{\delta \xi^{\mu\nu}} = \epsilon^A_\mu \epsilon_{\nu A} - g_{\mu\nu} = 0, \tag{2.6b}$$

an explicit expression for the Lagrangian multipliers can be obtained

$$\xi^{\mu\nu} = \epsilon^{\nu A}\left(\frac{\mathrm{D}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{\epsilon}^A_\mu} - \frac{\partial L}{\partial \epsilon^A_\mu}\right). \tag{2.7}$$

However, the same result could be obtained if $\sum_{0\leqslant\mu\leqslant\nu\leqslant 3} \xi^{\nu\mu}$ is used in (2.5). The symmetry of the orthonormal constraint must give us $\xi^{\mu\nu} = \xi^{\nu\mu}$, which in turn gives us the equation of motion for the tetrad expressed in terms of the general Lagrangian, whose form is completely generic for the moment. Using a more compact notation given by the variational derivative defined above, the equation of motion for the tetrad is succinctly expressed as

$$\frac{\delta L}{\delta \epsilon^A_{[\mu}} \epsilon^{\nu]A} = 0. \tag{2.8}$$

The requirement that L be a scalar can bring this into a more illuminating form. Consider an infinitesimal coordinate transformation $x^\mu \to \bar{x}^\mu = x^\mu + \xi^\mu$, the change of the Lagrangian should give 0,

$$\frac{\partial L}{\partial u^\mu}\delta_\xi u^\mu + \frac{\partial L}{\partial \epsilon^A_\mu}\delta_\xi \epsilon^A_\mu + \frac{\partial L}{\partial \dot{\epsilon}^A_\mu}\delta_\xi \dot{\epsilon}^A_\mu = \frac{\partial L}{\partial u^\mu} u^\nu \partial_\nu \xi^\mu - \frac{\partial L}{\partial \epsilon^A_\nu}\epsilon^A_\mu \partial_\nu \xi^\mu - \frac{\partial L}{\partial \dot{\epsilon}^A_\nu}\dot{\epsilon}^A_\mu \partial_\nu \xi^\mu = 0. \tag{2.9}$$

The infinitesimal change $\delta_\xi \Phi$ should only take the intrinsic transformation in the component form of Lie derivative, excluding the $-\xi^\nu \partial_\nu \Phi$ term, since it's the dependence of the Lagrangian on the tensors *themselves* that should fullfil the scalar requirement. It's also the reason for a sign difference for the latter two terms after the first equality. Eq. (2.9) combined with (2.8) and the definition (2.3), it can be derived that

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = p^\mu u^\nu - u^\mu p^\nu = 2p^{[\mu} u^{\nu]}. \tag{2.10}$$

For the equation of motion of the position $x^\mu(\tau)$ (or equivalently its conjugate momentum), consider a family of timelike curves $x^\mu(\tau, \eta)$, where η is a spacelike coordinate parametrizing the timelike curves each with its time coordinate τ. Among all the η's there is one that describes the trajectory of the spinning particles, i.e., the one extremizing the action holding the spin degree of freedom "fixed". The latter requirement must be translated into a covariant expression

$$\frac{\mathrm{D}}{\mathrm{d}\eta}\epsilon^A_\mu = w^\nu \nabla_\nu \epsilon^A_\mu = 0, \tag{2.11}$$

where $w^\mu = \partial x^\mu(\tau,\eta)/\partial\eta$, analogous to $u^\mu = \partial x^\mu(\tau,\eta)/\partial\tau$. Since ∂_τ and ∂_η are two basis vectors in this adapted frame, for any scalar function f, $[\partial_\tau, \partial_\eta] f = (u^\mu \nabla_\mu w^\nu - w^\mu \nabla_\mu u^\nu)\nabla_\nu f = 0$, implying

$$w^\nu \nabla_\nu u^\mu - u^\nu \nabla_\nu w^\mu = 0. \tag{2.12}$$

Now the variation with respect to η in a covariant manner gives

$$\delta_{x_\eta} S = \int \mathrm{d}\tau \delta\eta \left[\frac{\partial L}{\partial u^\mu}\frac{\mathrm{D}}{\mathrm{d}\eta}\left(\frac{\partial x^\mu}{\partial \tau}\right) + \underbrace{\frac{\partial L}{\partial \epsilon^A_\mu}\frac{\mathrm{D}}{\mathrm{d}\eta}\epsilon^A_\mu}_{=0} + \frac{\partial L}{\partial \dot{\epsilon}^A_\mu}\frac{\mathrm{D}\dot{\epsilon}^A_\mu}{\mathrm{d}\eta} \right]. \tag{2.13}$$

For the first term in the curly bracket, the use of (2.12) and an integration by part of $\mathrm{D}/\mathrm{d}\tau$ will give $-\dot{p}_\mu(\partial x^\mu/\partial\eta)$. For the last term, notice $\dot{\epsilon}^A_\mu = \mathrm{D}\epsilon^A_\mu/\mathrm{d}\tau$, the use of $[\nabla_\mu, \nabla_\nu]\,\omega_\sigma = R_{\mu\nu\sigma}{}^\rho\omega_\rho$ and the algebraic symmetry of the Riemann tensor will give, together with the first term

$$\delta_{x_\eta} S = \int \mathrm{d}\tau \left(-\dot{p}_\mu + \frac{\partial L}{\partial \dot{\epsilon}^A_\nu} R^\rho{}_{\nu\sigma\mu} u^\sigma \epsilon^A_\rho \right) \frac{\partial x^\mu}{\partial \eta} \delta\eta. \tag{2.14}$$

Making this variation vanish for arbitrary $\delta_\eta x^\mu \equiv (\partial x^\mu/\partial\eta)\delta\eta$ gives the equation of motion for the linear momentum p_μ. Together with the equation for the spin, the equation of motion of a spinning test particle in curved spacetime reads

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = 2p^{[\mu}u^{\nu]}, \tag{2.15a}$$

$$\frac{\mathrm{D}p_\mu}{\mathrm{d}\tau} = \frac{1}{2} S^{\rho\nu} R_{\rho\nu\sigma\mu} u^\sigma. \tag{2.15b}$$

Unlike the case for monopolar particle, the presence of the spin and its coupling with spacetime curvature prevent the momentum from being parallel transported. And the conjugate momentum is not necessarily proportional to the particle's 4-velocity with its mass being the proportion constant, as indicated by (2.15a). Eq. (2.15a) describes, along with a spin supplementary condition which specifies the point with respect to which the angular momentum is measured, the precession of spin in curved spacetime. Moreover, this derivation shows that the notion of variation, which is at the heart of the first law, is nothing but the first order differential with respect to an auxillary parameter. Such parameter is often fictitious, but the above example is an exception. It's a real spacelike coordinate. And it's exactly the non-commutivity of the covariant derivatives along this auxillary spacelike direction and the "time" direction for the continuously parametrized timelike curves that gives rise to the Riemann curvature in the final equation of motion.

2.2 Tulczyjew's reduction

The skeletonization amounts to an ansatz for the multipolar expansion of the SEM tensor T^{ab} describing a matter content with compact support

$$T^{ab}(x) = \sum_{l=0}^{\infty} \nabla_{i_1} \ldots \nabla_{i_l} \int_\gamma \mathcal{T}^{abi_1 i_2 \ldots i_l}(y)\delta_4(x, y(\tau))\,\mathrm{d}\tau. \tag{2.16}$$

The integration is done along a representative worldline. The spirit of skeletonization is depicted on the right side of Fig 1. The form of such an ansatz is motivated from the multipole expansion of the gravitational potential sourced by a general mass density $\rho(t, \vec{x})$ through Poisson equation. Physical interpretation of (2.16) is also drawn analogously: it describes a point particle endowed with multipole structures, whose subsequent orders perturbatively encode more and more information about the matter content. $\mathcal{T}^{abi_1\dots i_l}$, symmetric for ab, is the (unreduced) lth moment, while $\delta_4(x, y(\tau))$ is the invariant Dirac delta distribution restricting the matter content on the worldline $y(\tau) \in \gamma$ only. For practical purposes, the expansion is often truncated at a desired order as an effective description, such as when the characteristic size of the binary constituents are much less than their separation and when the GWs generated by such binary is observed at a distance much further than their separation.

However, the form of the multipole moments are still completely general, and their degrees of freedom should be reduced by the constraint of local energy momentum conservation $\nabla_b T^{ab} = 0$. Such a reduction is called Tulczyjew's reduction. This procedure will motivate the definition of linear momentum, spin tensor and quadrupole tensor, give the SEM tensor expressed in terms of them and allow to derive their equation of motion. This subsection reviews how it is done at dipolar order and demonstrates that the results are the same as in the general Lagrangian approach. A generalization to quadrupole order which has been done in [52] is only sketched, since the spirit is quite similar but the technicalities are too involved to bring any new insight.

The constraint $\nabla_b T^{ab} = 0$ on the multipole ansatz expanded to dipolar order gives

$$\begin{aligned} \nabla_b T^{ab}(x) &= \nabla_b \int_\gamma \mathcal{T}^{ab}(y)\delta_4(x, y(\tau))\,\mathrm{d}\tau + \nabla_b\nabla_c \int_\gamma \mathcal{T}^{abc}(y)\delta_4(x, y(\tau))\,\mathrm{d}\tau \\ &= \int_\gamma t^a\delta_4\,\mathrm{d}\tau + \nabla_b \int_\gamma t^{ab}\delta_4\,\mathrm{d}\tau + \nabla_b\nabla_c \int_\gamma t^{abc}\delta_4\,\mathrm{d}\tau = 0. \end{aligned} \tag{2.17}$$

On the second line, the arguments are understood as the same as the first, and the integrands $t^{abi_1\dots i_l}$ are in the so-called normal forms (to be reviewed in appendix A), meaning $t^{abi_1\dots i_l} = t^{ab(i_1\dots i_l)}$ and $u_{i_k} t^{ab(i_1\dots i_l)} = 0$ for $k = 1, 2, \dots, l$, u^a being the normalized 4-velocity of the particle along the worldline. According to the Tulczyjew's first theorem, such a normal form re-expansion always exist to a given multipole order. The general formula (A.6) reviewed in the appendix A applied to (2.17) yields

$$t^a = -\left(\mathcal{T}^{au} - (\mathcal{T}^{auu})^{\cdot} + 2\mathcal{T}^{a(cu)}\dot{u}_c\right)^{\cdot} + R_{bce}{}^{a}\left(\mathcal{T}^{eu\hat{c}}u^b - \frac{1}{2}\mathcal{T}^{e(\hat{b}\hat{c})}\right) \tag{2.18a}$$

$$t^{ab} = \mathcal{T}^{a\hat{b}} - 2(\mathcal{T}^{a(cu)})^{\cdot}h^b{}_c - \mathcal{T}^{auu}\dot{u}^b \tag{2.18b}$$

$$t^{abc} = \mathcal{T}^{a(\hat{b}\hat{c})}, \tag{2.18c}$$

where $\mathcal{T}^{a_1\dots u\dots a_n} \equiv \mathcal{T}^{a_1\dots a_k\dots a_n}u_{a_k}$, and $\mathcal{T}^{a_1\dots\hat{a}_k\dots a_n} \equiv \mathcal{T}^{a_1\dots b\dots a_n}h^{a_k}{}_b$, with $h_{ab} = g_{ab} + u_a u_b$ being the projection tensor. An orthogonal decomposition with respect to u^a can be performed on the unreduced multipole moments

$$\mathcal{T}^{ab} = mu^a u^b + 2m^{(a}u^{b)} + m^{ab} \tag{2.19a}$$

$$\mathcal{T}^{abc} = u^a u^b n^c + 2u^{(a}n^{b)c} + n^{abc} + o^{ab}u^c, \tag{2.19b}$$

where m is for now just the scalar field $\mathcal{T}^{uu}$ defined on the worldline, $m^{ab} = m^{(ab)}$, $n^{abc} = n^{(ab)c}$, $o^{ab} = o^{(ab)}$ and all the m's and n's are orthogonal to u^a with all their indices. By Tulczyjew's second theorem reviewed in appendix A, the t's in (2.18) all vanish as they are in normal form in the expansion for the vanishing $\nabla_b T^{ab}$. Plugging (2.19) into (2.18) and setting (2.18c) to 0 gives, by contracting the resulted equation with u^a, $n^{(bc)} = 0$ so $n^{ab} = n^{[ab]}$, and $n^{a(bc)} = 0$. The latter along with $n^{abc} = n^{(ab)c}$ yields $n^{abc} = 0$, simplifying a bit the remaining calculation. Setting (2.18b) to 0 gives

$$m^{ab} + m^b u^a = -h^b{}_c \left(o^{ac} + u^a n^c + n^{ac}\right)^{\cdot}. \tag{2.20}$$

Contracting with $h^c{}_a h^d{}_b$ on both sides and making use of $m^{cd} = m^{(cd)}$ gives

$$h^a{}_c h^b{}_d \dot{\sigma}^{cd} = 0, \quad \text{where } \sigma^{ab} \equiv u^{[a} n^{b]} + n^{ab}. \tag{2.21}$$

With the anti-symmetric σ^{ab} so defined, setting (2.18a) to 0 gives

$$\left(mu^a + m^a + u^a n^b \dot{u}_b + n^{ab}\dot{u}_b - \dot{o}^{ab} u_b\right)^{\cdot} = -\frac{1}{2} R_{cbd}{}^a \left(2\sigma^{dc} u^b + \sigma^{bc} u^d\right). \tag{2.22}$$

Define the left hand side as $\dot{p}^a$. For the right hand side, rename the dummy indices

$$\text{r.h.s. of (2.22)} = \frac{1}{2}(R_{bce}{}^a + 2R_{ecb}{}^a) u^e \sigma^{bc} \equiv \frac{1}{2} I_{bce}{}^a u^e \sigma^{bc} = \frac{1}{2}(3R_{bce}{}^a + 2R_{ebc}{}^a) u^e \sigma^{bc}. \tag{2.23}$$

The alternative form for $I_{bce}{}^a$ after the last equal sign is realized by the algebraic symmetries of Riemann tensor. Adding the two expressions for $I_{bce}{}^a$ and dividing it by 2 yields, making use of the anti-symmetry of σ^{bc},

$$\dot{p}^a = \frac{1}{2} R_{bce}{}^a (2\sigma^{bc}) u^e. \tag{2.24}$$

Comparing with (2.15b), the σ^{ab} defined in (2.21) should be nothing but half of the spin tensor. It remains to be checked that whether the spin tensor and linear momentum defined from the unreduced multipole moment as above satisfies the other equation of motion (2.15a) for the spin tensor, or in this context

$$\dot{\sigma}^{ab} = p^{[a} u^{b]}. \tag{2.25}$$

A direct calculation from the definition of p^a defined from the left hand side of (2.22) gives

$$p^{[a} u^{b]} = m^{[a} u^{b]} - u^{[a} n^{b]c} \dot{u}_c + u^{[a} \dot{o}^{b]c} u_c. \tag{2.26}$$

Anti-symmetrizing (2.20) to replace $m^{[a} u^{b]}$ for the first term produces a term to cancel the last term $u^{[a} \dot{o}^{b]c} u_c$. The remaining terms give

$$p^{[a} u^{b]} = \dot{n}^{ab} + u^{[a} \dot{n}^{b]} + \dot{u}^{[a} n^{b]} - u^{[c} \dot{n}^{b]c} u_c - u^{[a} n^{b]c} \dot{u}_c = \dot{\sigma}^{ab} - u^{[a} \left(n^{b]c} u_c\right)^{\cdot}, \tag{2.27}$$

where the first three terms gives $\dot{\sigma}^{ab}$ from definition and Leibniz rule, and the bracketed derivative term vanishes since n^{ab} is othogonal to u^a. Hence (2.24) and (2.25) are derived as the equations of motion equivalent to (2.15) for a skeletonized dipolar particle in curved spacetime. And with the p^a and $S^{ab} = 2\sigma^{ab}$ defined before, the expansion of (2.16) to $l = 1$ (dipolar order) can be expressed as

$$T^{ab} = \int_\gamma u^{(a} p^{b)} \delta_4 \, \mathrm{d}\tau + \nabla_c \int_\gamma u^{(a} S^{b)c} \delta_4 \, \mathrm{d}\tau. \tag{2.28}$$

To summarize, Tulczyjew's reduction amounts to a four-step algorithm:

1. Write down the skeleton ansatz (2.16) to the desired order. And bring $\nabla_b T^{ab} = 0$ to normal form. By Tulczyjew's second theorem, every integrand in normal form vanishes.
2. Perform an orthogonal decomposition of the unreduced multipole moments $\mathcal{T}^{abi_1 \ldots i_l}$ with respect to the 4-velocity of the particle. Plug it into the constraint imposed by the vanishing of the normal form in the previous step.
3. Motivate definitions of the reduced multipole moments and obtain their equations of motion.
4. Inserting the expressions of the reduced multipole moments back into the ansatz to get the reduced SEM tensor.

For quadrupolar particle, the Lagrangian approach is used in [53], and Tulczyjew's reduction is done in [52]. The calculation of the latter is rather involved, but the spirit aligns with the above four-step algorithms. To quote the results

$$\dot{p}^a = \frac{1}{2} R_{bcd}{}^a S^{bc} u^d - \frac{1}{6} J^{bcde} \nabla^a R_{bcde}, \tag{2.29a}$$

$$\dot{S}^{ab} = 2p^{[a} u^{b]} + \frac{4}{3} R_{edc}{}^{[a} J^{b]cde}. \tag{2.29b}$$

$$T^{ab} = \int_\gamma \left[u^{(a} p^{b)} + \frac{1}{3} R_{cde}{}^{(a} J^{b)cde} \right] \delta_4 \, \mathrm{d}\tau + \nabla_c \int_\gamma u^{(a} S^{b)c} \delta_4 \, \mathrm{d}\tau - \frac{2}{3} \nabla_c \nabla_d \int_\gamma J^{c(ab)d} \delta_4 \, \mathrm{d}\tau. \tag{2.30}$$

J^{abcd} is the reduced quadrupole moment, which encodes part of the information about the internal structure as well as a spin induced contribution of the particle moving along its worldline. It possesses the same algebraic symmtries as those of the Riemann tensor. The result for the SEM tensor is used to prove that the two particles in binary follow helical Killing trajectories, a result appearing in previous literature but for the first time proven in [39] for the general case of arbitrary mass ratio of spinning particles with internal structures.

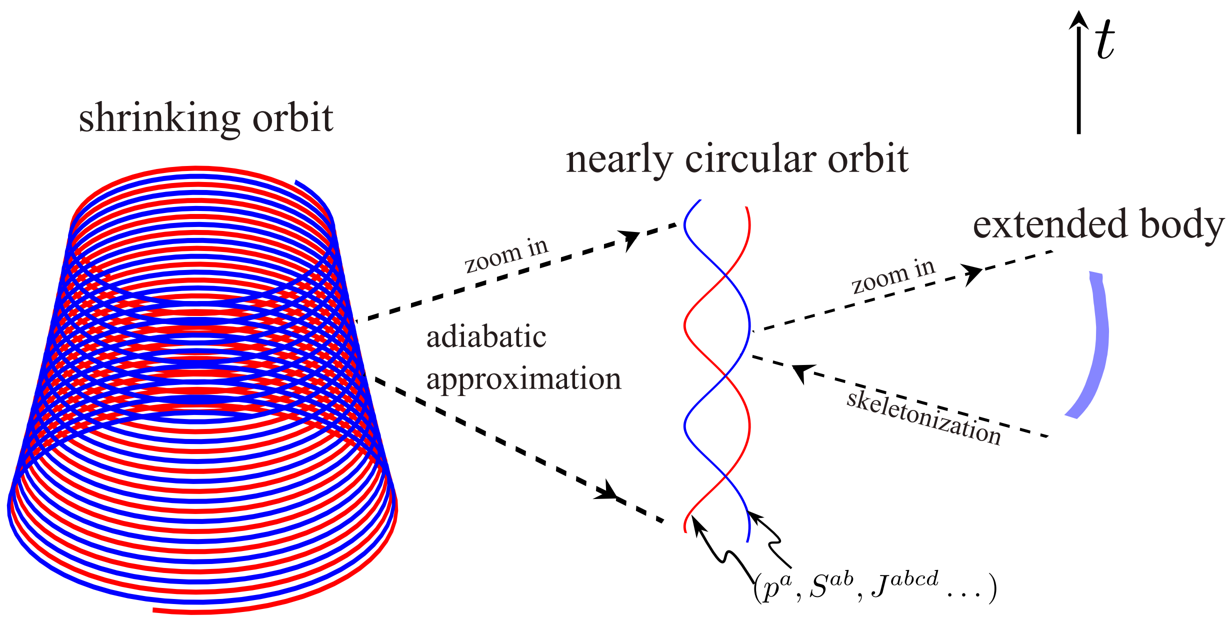


Figure 1: Illustration of adiabatic approximation and skeletonization. On the left is the realistic shrinking inspiral undergoing GW backreaction. The adiabatic approximation amounts to treating the orbit as circular and stable, which captures the main feature of realistic astrophysical binary systems by the quantitative analysis in the main text in section 3. On the right, the sprit of skeletonization is presented: an extended body swiping out a world-tube, viewed "far-away", effectively reduced to a particle travelling on a worldline endowed with (Tulczyjew-reduced) multipole moments. One spatial dimension is supressed, namely the "z-axis".

3 Helical Killing vector field

The picture under consideration is two point particles with internal structures in circular orbit. The circular orbit is in turn translated in the existence of a helical Killing vector (HKV) field

$$k^a = (\partial_t)^a + \Omega(\partial_\phi)^a, \tag{3.1}$$

where $(\partial_t)^a$ and $(\partial_\phi)^a$ are the basis vectors generating time translation and rotation of azimuthal angle at asymptotic infinity. Several justifications have to be made before continuing: (i)The spacetime is sourced by the two point particles endowed with internal structures in skeleton formalism, understood in a regularized sense. (ii) Strictly speaking spacetimes with HKV cannot be asymptotically flat. It's only under some approximate schemes can this make sense, such as to a certain PN order. Another approach is to solve a set of field equations in spacetime with spatial conformal flatness[54, 55]. (iii) In reality the orbit does not necessarily have to be circular, and it will undergo backreaction by the

GWs emitted and therefore shrink to merger. However, for systems of realistic astrophysical interest, it's often the case that the orbit efficiently circularizes and stays near circular for a long time. It means the timescales of circularization, the orbital period and the final merger are all much less than the time scale for which the orbital eccentricity is nearly 0. Hence the consideration of stable circular orbit as adiabatic approximation, as is pictorially represented on the left of Fig 1, is well motivated. The quantitative discussion for a binary system of masses m_1 and m_2, with semi-major axis a and eccentricity e, is presented in chapter 4 of [17], which is reviewed here. At leading PN order

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{64}{5}\frac{\mu m^2}{a^3}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), \tag{3.2a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{304}{15}\frac{\mu m^2}{a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right), \tag{3.2b}$$

where $m = m_1 + m_2$ is the total mass and $\mu = m_1 m_2/m$ is the reduced mass. The equations suggest both the shrinking and circularization of the orbit because of the minus signs in the rate of change. Equation (3.2b) is saying that the smaller the eccentricity, the slower its rate of change. In particular, for nearly circular orbit $e \simeq 0$, $\dot{e} \simeq 0$, meaning the orbit stays nearly circular. In particular the relation $a(e)$ can be analytically integrated from (3.2), expressed in terms of a function $g(e)$ defined to be $g(e) \equiv e^{12/19}/(1-e^2)\left[1+(121/304)e^2\right]^{870/2299}$, as

$$a(e) = a_0\frac{g(e)}{g(e_0)}. \tag{3.3}$$

For example, for the Hulse-Taylor pulsar, its present binary geometric parameter values are $a_0 \simeq 2\times 10^9$ m and $e_0 \simeq 0.617$. By the time the semi-major axis becomes hundreds of the radius of a typical neutron star $a \simeq 10^3$ km, the equations above implies $e \sim 6\times 10^{-6}$. That means for a typical astrophysical binary system, the orbit is almost perfectly circular when the separation of the binary constituents is still much larger than their sizes, meaning the circularization process is often quite efficient.

With these considerations, this section recapitulates the results in [39], i.e., (i) the worldlines of the two particles follow the integral lines of the HKV and (ii) the Tulczyjew reduced multipole moments, namely the linear momentum, the spin tensor and quadrupole tensor are all Lie-dragged along the HKV. These results will be used to derive the first law later. Technical details of the proofs, with minor modifications and refinements to [39], is relegated to appendix B. Here only intuitive physical arguments and interpretations will be provided.

The HKV $k^a = (\partial_t)^a + \Omega(\partial_\phi)^a$, by definition aligns with the 4-velocity of an observer corotating with the binary wherever it is timelike. Hence, on the worldline, we have

$$k^a\Big|_\gamma = z u^a. \tag{3.4}$$

(The statements will be true for both particles, hence the index i is omitted for brevity.) The non-trivial part, is that the proportionality scalar z is actually a constant along the worldline, i.e., $\dot{z} = 0$. This can be seen by arguing that the factor z is actually the redshift factor between the light emitted from one of the particle to the light received by an observer sitting at the symmetry axis of rotation. Suppose a beam of light is emitted by an observer riding with the particle. The energy she measures will be $E_e = -u^a p_a$. Since the momentum of the photon p^a follows null geodesic and k^a is a Killing vector, the scalar $k^a p_a$ is a constant of motion, as can be easily seen by a one line calculation:

$$p^a \nabla_a (k_b p^b) = p^a p^b \nabla_a k_b + k_b p^a \nabla_a p^b = 0, \tag{3.5}$$

using Killing's equation and geodesic equation. If an observer is sitting at the symmetry axis of rotation, the vector $(\partial_\phi)^a$ vanishes there, only the vector $(\partial_t)^a$ generates his time evolution. The ratio of energies measured upon reception and emission is

$$\frac{E_r}{E_e} = \frac{-k^a p_a\big|_S}{-u^a p_a\big|_\gamma} = \frac{k^a p_a\big|_\gamma}{u^a p_a\big|_\gamma} = z. \tag{3.6}$$

The second equality makes use of the fact that $k^a p_a$ is conserved to "trace its value back to emission". And the third equality follows from (3.4). Thus the name "redshift factor" is justified and it is dubbed Detweiler's redshift observable in the literature. Since stable circular orbit is considered here, it should not change. A more mathematical proof is provided in [39] and reviewed in appendix B with minor changes to make the argument more rigorous and clear.

Then there is the result that

$$\mathcal{L}_k p^a = 0, \qquad \mathcal{L}_k S^{ab} = 0, \qquad \mathcal{L}_k J^{abcd} = 0. \tag{3.7}$$

The proof makes use of the fact $\mathcal{L}_k T^{ab} = 0$ from Einstein equation, Tulczyjew's theorems concerning normal form expansions, and a (3+1)-decomposition of tensors possessing algebraic symmetries of Riemann tensor such as J^{abcd}. Again the details are relegated to appendix B. Since $k^a\big|_\gamma = z u^a$ with $\dot{z} = 0$, equivalently $\mathcal{L}_u p^a = \mathcal{L}_u S^{ab} = \mathcal{L}_u J^{abcd} = 0$. The physical interpretation of this result is that the reduced multipole moments characterizing the skeletonized particle is not altered by the geometric flow generated by its own time evolution, which should be expected by physical intuition.

4 Mathematical and theoretical preliminaries of the first law

The idea of combining differential forms and symplectic geometry in field theory, intiated by Iyer and Wald[25, 26], has a wide range of applications. It is reviewed in this section. As a consequence, the first law of BH mechanics first found by BCH [24] is rederived using the new method, which brings additional insight and opens up new directions to explore. Specifically, such a strategy will be applied to the case of GR involving an arbitrary matter source of compact support in Cauchy surfaces of a globally hyperbolic spacetime, to derive the key identity in Ref. [49]

$$\delta H_\xi = \frac{1}{2}\int\limits_\Sigma \xi^a \epsilon_{abcd} T^{ef} \delta g_{ef} - \delta \int\limits_\Sigma \epsilon_{abcd} T^{ae} \xi_e. \tag{4.1}$$

Here ξ^a is a vector field in spacetime generating a diffeomorphism. H_ξ is the corresponding conserved quantity associated with it if it is a symmetry, which is in general *not* what the literatures refer to as the Noether charge as will be clarified later. In particular, if there exists a dynamics generated by ξ^a, H_ξ will be the corresponding Hamiltonian. The identity relates the variation of H_ξ with two integrals involving the matter content over an arbitrary 3-dimensional spacelike hypersurface Σ. This identity will be further applied to the specific scenario that the matter content T^{ab} being that of the point particle binary system in gravitational skeleton formalism. In this way, the first laws relating the variations of global quantities with those of local ones are derived in the next section.

4.1 General approach

In this subsection we shall introduce the general method of differential forms for classical field theory coupled to gravity presented in a series of references[56, 25, 26].

In the most generic setting, consider a Lagrangian n-form $\boldsymbol{L}$ in a smooth oriented spacetime manifold M, depending on dynamical fields collectively denoted as $\phi = (g_{ab}, \psi)$ consisting of the Lorentzian metric g_{ab} and a set of matter fields ψ. Let $\mathring{\nabla}$ denote a fixed (in the sense of when a variation is performed), globally defined derivative operator. In addition, the Lagrangian should also depend on an arbitrary number of symmetrized derivatives of the dynamical fields, as well as some "background fields" $\mathring{\gamma}$, for which, the Riemann tensor $\mathring{R}_{abcd}$ associated with $\mathring{\nabla}$ is an example. In other words, we are considering a Lagrangian n-form with functional dependence

$$\boldsymbol{L} = \boldsymbol{L}(g_{ab}, \mathring{\nabla}_{(c_1} \dots \mathring{\nabla}_{c_k)} g_{ab}, \psi, \mathring{\nabla}_{(a_1} \dots \mathring{\nabla}_{a_l)} \psi, \mathring{\gamma}). \tag{4.2}$$

Moreover, the Lagrangian should be diffeomorphism covariant, meaning for any diffeomorphism $f : M \to M$, $\boldsymbol{L}(f^*\phi) = f^*\boldsymbol{L}(\phi)$, where f^* is the map induced on tensor fields. Ref. [26] shows that such a Lagrangian can always be rewritten as

$$\boldsymbol{L} = \boldsymbol{L}(g_{ab}, R_{abcd}, \nabla_{(e_1} \dots \nabla_{e_m)} R_{abcd}, \psi, \nabla_{(a_1} \dots \nabla_{a_l)} \psi), \tag{4.3}$$

where ∇_a is the metric compatible derivative operator and R_{abcd} is the Riemann tensor associated with it. There is no more explicit dependence of the Lagrangian on the background fields. Such a re-expression of $\boldsymbol{L}$ guarantees that in its variation

$$\delta \boldsymbol{L} = \boldsymbol{E}_\phi \delta\phi + \mathrm{d}\boldsymbol{\Theta}(\phi, \delta\phi), \tag{4.4}$$

the symplectic potential $(n-1)$-form $\boldsymbol{\Theta}(\phi, \delta\phi)$ constructed from dynamical fields and their variations, always exists and can be expressed in a covariant canonical form, with dependence on the dynamical fields and their variations. Notice that $\boldsymbol{\Theta}(\phi, \delta\phi)$ is ambigiuous for an inclusion of boundary $(n-1)$-form $\delta\boldsymbol{\mu}$ coming from the ambiguity of the Lagrangian, and an exact form $\mathrm{d}\boldsymbol{Y}$ for an $(n-2)$-form $\boldsymbol{Y}$. For the moment its explicit form is not of too much importance for deriving the first law of BH. And its ambiguity should not have any impact on subsequent applications as is illustrated later. The first term in the variation can be further written as

$$\boldsymbol{E}_\phi \delta\phi = \boldsymbol{E}_g^{ab} \delta g_{ab} + \boldsymbol{E}_\psi \delta\psi. \tag{4.5}$$

The $\boldsymbol{E}$'s appearing here are the equation of motion forms. They are n-forms if the indices dual to those of dynamical fields are removed. Contraction with tensor indices and a summation of matter fields should be understood for the second term. When the equations of motion for the dynamical fields are satisfied, $\boldsymbol{E}_\phi = 0$, $\boldsymbol{E}_g^{ab} = 0$ and $\boldsymbol{E}_\psi = 0$.

Let ξ^a be a vector in M generating a diffeomorphism. The Noether current $(n-1)$-form associated with it is defined as

$$\boldsymbol{J}[\xi] \equiv \boldsymbol{\Theta}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \boldsymbol{L}, \tag{4.6}$$

where $\xi \cdot \boldsymbol{L}$ denotes the interior product of ξ^a with $\boldsymbol{L}$ (see appendix C.2). Notice that the Lie derivative of the dynamical fields $\mathcal{L}_\xi \phi$ can play the role of a variation $\delta\phi$ as a vector in the configuration manifold $\mathcal{F}$ of the dynamical fields consisting of all possible field configurations, in which the one satisfying the equations of motion is just one point. The resemblance of the definition (4.6) with the vector expression in flat spacetime field theory is mentioned in appendix C.1. The exterior differential of $\boldsymbol{J}[\xi]$ reads

$$\begin{aligned} \mathrm{d}\boldsymbol{J}[\xi] &= \mathrm{d}\boldsymbol{\Theta}(\phi, \mathcal{L}_\xi \phi) - \mathrm{d}(\xi \cdot \boldsymbol{L}) \\ &= \mathrm{d}\boldsymbol{\Theta}(\phi, \mathcal{L}_\xi \phi) - \mathcal{L}_\xi \boldsymbol{L} + \xi \cdot \mathrm{d}\boldsymbol{L} \\ &= \mathcal{L}_\xi \boldsymbol{L} - \boldsymbol{E}_\phi \mathcal{L}_\xi \phi - \mathcal{L}_\xi \boldsymbol{L} \\ &= -\boldsymbol{E}_\phi \mathcal{L}_\xi \phi. \end{aligned} \tag{4.7}$$

The second line made use of the Cartan's magic formula $\mathcal{L}_\xi \boldsymbol{\omega} = \mathrm{d}(\xi \cdot \boldsymbol{\omega}) + \xi \cdot \mathrm{d}\boldsymbol{\omega}$ for a general p-form $\boldsymbol{\omega}$. The third line made use of (4.4) with $\mathcal{L}_\xi$ playing the role of $\delta\phi$, and the vanishing of $\mathrm{d}\boldsymbol{L}$ due to the rank n of the form $\boldsymbol{L}$. Meanwhile, it can be shown that there exists an $(n-1)$-form $\boldsymbol{C}_a \xi^a$, for which $\boldsymbol{C}_a$ is called the constraint, such that its exterior derivative gives $-\boldsymbol{E}_\phi \mathcal{L}_\xi \phi$ in the final result[26]. Furthermore, if the equation of motion is satisfied, $\boldsymbol{C}_a$ vanishes as well. It means that there exists a Noether charge $(n-2)$-form $\boldsymbol{Q}$ constructed locally from the fields and their variations, such that

$$\boldsymbol{J}[\xi] = -\boldsymbol{C}_a \xi^a + \mathrm{d}\boldsymbol{Q}. \tag{4.8}$$

When the fields are on shell, $\boldsymbol{J}[\xi]$ is closed and free of the first term, hence exact and equal to $\mathrm{d}\boldsymbol{Q}$. It should be taken with care that when matter is coupled to gravity, e.g., in GR, the vanishing of $\boldsymbol{C}_a$ does not imply that its contribution from gravity (i.e., the metric) and matter separately vanishes. In fact, their cancellation yields the Einstein equation (just as the fact that Einstein equation being true does not imply the vanishing of its both sides). This will be taken with caution in the next subsection. However, in this subsection, in the interest of deriving the first law for BH, only spacetime is concerned. Hence, whenever the equation of motion is satisfied one can take the Noether current $(n-1)$-form to be exact. In particular, the fact that $\mathrm{d}\boldsymbol{J}[\xi] = 0$ when $\boldsymbol{E}_\phi = 0$ is basically the statement of Noether theorem. Heuristically, it's a generalization of the flat spacetime form-free version reviewed in appendix C.1.

Consider two independent variations $\delta_1\phi$ and $\delta_2\phi$, formally speaking two linearly independent vectors in the configuration manifold $\mathcal{F}$. The symplectic current $(n-1)$-form $\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi)$ is defined as

$$\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) \equiv \delta_1\boldsymbol{\Theta}(\phi, \delta_2\phi) - \delta_2\boldsymbol{\Theta}(\phi, \delta_1\phi). \tag{4.9}$$

In a globally hyperbolic spacetime, its integration over a Cauchy surface $\mathcal{C}$ is well defined and free of the ambiguities of the symplectic potential $\boldsymbol{\Theta}$ if proper asymptotic behavior of the dynamical fields and their variations are fulfilled[26], which is assumed throughout the present work. The integration

$$\Omega = \int_{\mathcal{C}} \boldsymbol{\omega}(\delta_1\phi, \delta_2\phi) \tag{4.10}$$

gives the so-called presymplectic form Ω, which after quotienting the redundent degrees of freedom from gauge invariance by diffeomorphism covariance, generate the symplectic form and the corresponding phase space for field dynamics

$$\Omega = \Omega_{AB}(\delta_1\phi)^A(\delta_2\phi)^B, \tag{4.11}$$

where A, B are abstract indices for tensors in the field configuration manifold. It can be easily seen that Ω_{AB} is indeed anti-symmetric. In particular, consider $\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi)$, where $\delta\phi$ is an arbitrary variation independent of $\mathcal{L}_\xi\phi$ but satisfies the equation of motion. If ξ^a generates a dynamical evolution for the fields and the associated Hamiltonian H_ξ exists, then the symplectic geometry for classical field theory generalized from classical mechanics yields[56]

$$\int_{\mathcal{C}} \boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi) = \delta H_\xi. \tag{4.12}$$

In the case of ξ^a held fixed, $\delta\xi^a = 0$, when the fields are on shell, $\boldsymbol{E}_\phi = 0$, an explicit calculation of $\delta\boldsymbol{J}[\xi]$ from (4.6) gives

$$\begin{aligned}\delta\boldsymbol{J}[\xi] &= \delta\boldsymbol{\Theta}(\phi, \mathcal{L}_\xi\phi) - \xi\cdot\delta\boldsymbol{L}\\ &= \delta\boldsymbol{\Theta}(\phi, \mathcal{L}_\xi\phi) - \xi\cdot(\boldsymbol{E}_\phi\delta\phi + \mathrm{d}\boldsymbol{\Theta}(\phi, \delta\phi))\\ &= \delta\boldsymbol{\Theta}(\phi, \mathcal{L}_\xi\phi) - \mathcal{L}_\xi\boldsymbol{\Theta}(\phi, \delta\phi) - \mathrm{d}(\xi\cdot\boldsymbol{\Theta}(\phi, \delta\phi))\\ &= \boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi) + \mathrm{d}(\xi\cdot\boldsymbol{\Theta}(\phi, \delta\phi)),\end{aligned} \tag{4.13}$$

where Cartan's magic formula and the on-shell condition are used in the third line. Then one has

$$\delta H_\xi = \delta \int_{\mathcal{C}} \mathrm{d}\boldsymbol{Q}[\xi] - \int_{\mathcal{C}} \mathrm{d}(\xi \cdot \boldsymbol{\Theta}(\phi, \delta\phi)) = \delta \int_{\partial\mathcal{C}} \boldsymbol{Q}[\xi] - \int_{\partial\mathcal{C}} \xi \cdot \boldsymbol{\Theta}(\phi, \delta\phi). \tag{4.14}$$

If there exists an $(n-2)$-form $\boldsymbol{B}$ such that

$$\int_{\partial\mathcal{C}} \xi \cdot \boldsymbol{\Theta}(\phi, \delta\phi) = \delta \int_{\partial\mathcal{C}} \xi \cdot \boldsymbol{B}, \tag{4.15}$$

then

$$H_\xi = \int_{\partial\mathcal{C}} \boldsymbol{Q}[\xi] - \xi \cdot \boldsymbol{B}. \tag{4.16}$$

In particular, when $\mathcal{C}$ has no interior boundary and $\partial\mathcal{C}$ is taken to be a $(n-2)$ surface at spatial infinity, (4.16) coincides with the ADM charges. In particular, in GR where the Lagrangian is specified to the Einstein-Hilbert one, such a 3-form $\boldsymbol{B}$ exists.

4.2 The first law of black hole mechanics

Armed with all this prerequisite of form expression of Noether theorem and symplectic geometry theory, as in Ref. [26] we shall now derive the first law for arbitrary perturbations of an axisymmetric asymptotically flat stationary BH solution of a generic diffeomorphism covariant theory of gravity involving a metric in spacetime dimension n. When restricted to GR, it agrees with the one found by BCH[24]. Such a derivation not only gives a local geometrical prescription of BH entropy in a broader context, but also demonstrates the utility of the differential form method, paving its way to applications to other scenarios.

Consider a general diffeomorphism covariant theory of gravity without matter in n spacetime dimension. Suppose there is a stationary asymptotically flat black hole (BH) solution admitting an axisymmetric Killing vector

$$\chi^a = t^a + \Omega_H \varphi^a, \tag{4.17}$$

where t^a and φ^a are vectors generating time translation and azimuthal rotations in spatial infinity. A summation of all the axisymmetric directions should be understood for the second term if $n > 4$. Ω_H's are the angular velocity of the BH at the outer horizon. This means that χ^a is the generator of its Killing horizon. In particular, it vanishes at the bifurcation Killing horizon which is an inner boundary of the Cauchy surface $\mathcal{C}$. In this case $\phi = g_{ab}$ and take ξ^a to beχ^a. The fact that χ^a is Killing, i.e., $\mathcal{L}_\chi \phi = 0$, leads to the vanishing of $\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\chi \phi)$. Eq. (4.14) applied in this case is

$$0 = \delta \int_{\mathcal{C}} \mathrm{d}\boldsymbol{Q}[\chi] - \int_{\mathcal{C}} \mathrm{d}(\chi \cdot \boldsymbol{\Theta}(\phi, \delta\phi)). \tag{4.18}$$

The Cauchy surface $\mathcal{C}$ has two boundaries: one is a topological two-sphere at spatial infinity $\mathbb{S}^2_\infty$ with "outward-pointing" normal vector, the other is the bifurcation Killing horizon $\mathscr{H}$ with "inward-pointing" normal vector. The linearity of $\boldsymbol{Q}[\chi]$ and Stokes's theorem applied for (4.18) yields

$$\delta \int_{\mathbb{S}^2_\infty} \boldsymbol{Q}[t] - \int_{\mathbb{S}^2_\infty} t \cdot \boldsymbol{\Theta}(\phi, \delta\phi) + \delta \int_{\mathbb{S}^2_\infty} \Omega_H \boldsymbol{Q}[\varphi] - \delta \int_{\mathscr{H}} \boldsymbol{Q}[\chi] = 0. \tag{4.19}$$

For the second term, there exists an $(n-2)$-form $\boldsymbol{B}$ that satisfies (4.15) with $\mathbb{S}^2_\infty$ playing the role of $\mathcal{C}$. For the third term, there is no $\varphi \cdot \boldsymbol{\Theta}$ term since φ^a is tangent to $\mathbb{S}^2_\infty$, hence orthogonal to its normal vector. For the fourth term, the minus sign is caused by the 'inward-pointing" normal vector of Σ, and χ^a vanishes at Σ. The first two terms give rise to the variation of the ADM mass and angular momentum

$$\delta M = \delta \int_{\mathbb{S}^2_\infty} \boldsymbol{Q}[t] - t \cdot \boldsymbol{B}, \tag{4.20a}$$

$$\delta J = - \int_{\mathbb{S}^2_\infty} \boldsymbol{Q}[\varphi]. \tag{4.20b}$$

Putting everything together,

$$\delta \int_{\mathscr{H}} \boldsymbol{Q}[\chi] = \delta M - \Omega_H \delta J. \tag{4.21}$$

For the left hand side, since $\boldsymbol{Q}[\chi]$ is locally constructed from the dynamical fields and χ as well as their symmetrized covariant derivatives, and is invariant for diffeomorphisms acting on $\mathscr{H}$. χ^a vanishes on $\mathscr{H}$. Successive use of the formula $\nabla_a \nabla_b \chi_c = R_{cbad}\chi^d$ for the Killing vector χ^a can express $\boldsymbol{Q}[\chi]$ in terms of $\nabla_a \chi_b$ only, which is actually proportional to the binormal ϵ_{ab} of $\mathscr{H}$ with the surface gravity κ being the proportionality constant, in the case of stationary background. Replacing $\nabla_a \chi_b$ everywhere in the final expression of $\boldsymbol{Q}[\chi]$ by ϵ_{ab} allows the construction of the Noether charge $\tilde{\boldsymbol{Q}}[\chi]$ with unit surface gravity. The ability to factorize out the surface gravity for stationary perturbations on the left hand side of (4.21) is demonstrated in [25], while the generalization to arbitrary perturbations is shown in [26]. Eventually

$$\delta M - \Omega_H \delta J = \frac{\kappa}{2\pi}\, \delta \left(2\pi \int_{\mathscr{H}} \tilde{\boldsymbol{Q}}[\chi] \right) = \frac{\kappa}{2\pi} \delta S. \tag{4.22}$$

In the last equality, the integral in the bracket is identified as the entropy of the stationary asymptotically flat BH. Up to now, the theory under consideration is completely general. Restricting to GR simply reproduces the result of the first law of BH thermodynamics found by BCH[24]. However, such a result has more profound implications: (i) the entropy of a stationary BH, in any diffeomorphism covariant theory of gravity involving a metric, is endowed with a local geometrical prescription as the integration of Noether charge $(n-2)$-form of the horizon Killing vector, at the bifurcation Killing horizon. (ii) A first law for stationary BH holds in the general setting of (i) even beyond GR, provided that the BH entropy is recognized as the scalar Noether charge. (iii) Such an analysis provides hints for a definition of entropy for dynamical BH, which is further explored in [26]. (iv) The recovery of the well-known BCH's first law in this line of reasoning suggests its applicability for other scenarios, such as for binary BHs and perfect fluid bodies[28], and point particle binary systems[29, 30, 49]. The application to the latter situation is the main interest of the present work.

4.3 Key variational formula

Consider the theory of GR in a globally hyperbolic spacetime, with an arbitrary matter source described by a SEM tensor T^{ab} with compact support in each spacelike hypersurface $\Sigma_{t(t\in\mathbb{R})}$ for an arbitrary foliation. Let ξ^a be a Killing vector kept fixed. For an arbitrary variation δ (with $\delta\xi^a = 0$) and the variation $\mathcal{L}_\xi$, the symplectic current 3-form, defined in (4.9) can be computed as

$$\begin{aligned}
\boldsymbol{\omega}(\phi,\delta\phi,\mathcal{L}_\xi\phi) &= \delta\boldsymbol{\Theta}(\phi,\mathcal{L}_\xi\phi) - \mathcal{L}_\xi\boldsymbol{\Theta}(\phi,\delta\phi) \\
&= \delta\boldsymbol{J}[\xi] + \xi\cdot\delta L - \xi\cdot\mathrm{d}\boldsymbol{\Theta}(\phi,\delta\phi) - \mathrm{d}(\xi\cdot\boldsymbol{\Theta}(\phi,\delta\phi)) \\
&= \delta\boldsymbol{J}[\xi] + \xi\cdot(\boldsymbol{E}_\phi\delta\phi + \mathrm{d}\boldsymbol{\Theta}(\phi,\delta\phi)) - \xi\cdot\mathrm{d}\boldsymbol{\Theta}(\phi,\delta\phi) - \mathrm{d}(\xi\cdot\boldsymbol{\Theta}(\phi,\delta\phi)) \\
&= \mathrm{d}\left(\delta\boldsymbol{Q}[\xi] - \xi\cdot\boldsymbol{\Theta}(\Psi\delta\Psi)\right) - \delta(\boldsymbol{C}_a\xi^a) + \xi\cdot\boldsymbol{E}_\phi\delta\phi.
\end{aligned} \tag{4.23}$$

At the second line, the definition of Noether current $(n-1)$-form and Cartan's magic formula are used. (4.4) is used in the third line. The last line replaces $\boldsymbol{J}[\xi]$ by the expression (4.8), with $\boldsymbol{Q}$ being the Noether charge 2-form. When the equation of motions are satisfied, including the one for the matter fields and Einstein equation, the latter two terms in the result vanish. Alternatively the symplectic current 3-form can also be calculated from a gravity-matter split. In this case, the Lagrangian 4-form is decomposed as $\boldsymbol{L} = \boldsymbol{L}_g + \boldsymbol{L}_m$, where the first term is the Einstein-Hilbert Lagrangian. The variation satisfies

$$\delta\boldsymbol{L}_g = \boldsymbol{E}_g^{ab}\delta g_{ab} + \mathrm{d}\boldsymbol{\Theta}_g(g,\delta g), \tag{4.24a}$$

$$\delta\boldsymbol{L}_m = \frac{1}{2}\boldsymbol{\epsilon}T^{ab}\delta g_{ab} + \boldsymbol{E}_m\delta\psi + \mathrm{d}\boldsymbol{\Theta}_m(\phi,\delta\phi). \tag{4.24b}$$

For the symplectic current $\boldsymbol{\omega} = \boldsymbol{\omega}_g(g,\delta g,\mathcal{L}_\xi g) + \boldsymbol{\omega}_m(\phi,\delta\phi,\mathcal{L}_\xi\phi)$, the first term $\boldsymbol{\omega}_g$ vanishes since ξ^a is Killing, in analogy with (4.18) for the black hole case. The calculation for $\boldsymbol{\omega}_m$ completely parallels (4.23), and the result is

$$\boldsymbol{\omega}_m = \mathrm{d}\left(\delta\boldsymbol{Q}_m[\xi] - \xi\cdot\boldsymbol{\Theta}_m(\phi,\delta\phi)\right) + \frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab} - \delta(\boldsymbol{C}_a^m\xi^a) + \xi\cdot\boldsymbol{E}_m\delta\psi. \tag{4.25}$$

The results for (4.25) and (4.23) are the same, with $\boldsymbol{E}_\phi = 0$, $\boldsymbol{C}_a = 0$, and $\boldsymbol{E}_m = 0$ as a consequence of the equations of motion being satisfied. Notice that $\boldsymbol{C}_a^m$ itself does not vanish. $\boldsymbol{C}_a^m = -\boldsymbol{C}_a^g$ is a consequence of Einstein equation. In this case, the matter constraint is shown to be[57]

$$\boldsymbol{C}^m_{a[cde]} = T^b{}_a\,\epsilon_{bcde}. \tag{4.26}$$

Eventually the equality implies

$$\mathrm{d}(\delta \boldsymbol{Q}_g[\xi] - \xi \cdot \boldsymbol{\Theta}_g(g, \delta g)) = \frac{1}{2}\xi \cdot \boldsymbol{\epsilon} T^{ab}\delta g_{ab} - \delta\left[(\xi^b T^a{}_b)\cdot \boldsymbol{\epsilon}\right]. \tag{4.27}$$

If there is no interior boundary for the spacelike hypersurfaces Σ of constant "time" (as is opposed to the black hole case done in the last subsection), and if there exists a 2-form $\boldsymbol{B}_g$ to factor out the δ for the second term on the left hand side, the integral formula for (4.27) is

$$\delta \int_{\partial\Sigma} \boldsymbol{Q}_g[\xi] - \xi \cdot \boldsymbol{B}_g = \frac{1}{2}\int_{\Sigma} \xi^a \epsilon_{abcd} T^{ef}\delta g_{ef} - \delta \int_{\Sigma} \xi^e T^a{}_e\,\epsilon_{abcd}. \tag{4.28}$$

In the absence of interior boundary for Σ, its only boundary is spatial infinity, then $\partial\Sigma$ is a topological 2-sphere with "infinite radius" $\mathbb{S}^2_\infty$. Since ξ^a is Killing, the left hand side of (4.28) will define the variation of the conserved quantity H_ξ associated with ξ^a for spacetime, as is seen by an asymptotically inertial observer (in particular it's the Hamiltonian, or ADM energy if ξ^a generates time translation for this asymptotically inertial observer). Hence we have[49]

$$\delta H_\xi = \frac{1}{2}\int_{\Sigma} \xi^a \epsilon_{abcd} T^{ef}\delta g_{ef} - \delta \int_{\Sigma} \xi^e T^a{}_e\,\epsilon_{abcd}, \tag{4.29}$$

the key variational formula (4.1) mentioned at the beginning of the section. The application of this formula to the physical setting laid in section 3 is of the most interest for the present work.

5 The first law: dipolar and quadrupolar order

5.1 Generalities

Before applying (4.29) to the specific cases to derive the first law, which is the goal of the current section, a few more general remarks have to be made. For the binary system composed of point particles described by gravitational skeleton formalism, take ξ^a in the general analysis in the previous section to be the HKV $k^a = (\partial_t)^a + \Omega(\partial_\phi)^a$. Then indeed the spacelike hypersurface Σ does not have interior boundary. And the left hand side of (4.28) is calculated in the same manner as (4.21)

$$\delta H_k = \delta\left(\int_{\Sigma} \boldsymbol{Q}_g[t] - t\cdot \boldsymbol{B}_g\right) - \Omega\,\delta\int_{\Sigma} \boldsymbol{Q}_g[\phi] = \delta M - \Omega\delta J. \tag{5.1}$$

Notice that strictly speaking a helical symmetric spacetime cannot be asymptotically flat. However in certain approximation regimes the asymptotic behavior of such a spacetime can indeed be treated as flat. It's in this situation and the fact that $(\partial_t)^a$ and $(\partial_\phi)^a$ are *asymptotically* Killing will it makes sense to talk about the ADM mass and ADM angular momentum (and their variations) in the above equation. Moreover, since $(\partial_t)^a$ and $(\partial_\phi)^a$ generate time translation and azimuthal rotation for an asymptotically inertial observer, the Komar mass and Komar angular momentum, defined as

$$M_K = 2\int\limits_{\partial\Sigma} \boldsymbol{Q}_g[t], \qquad J_K = -\int\limits_{\partial\Sigma} \boldsymbol{Q}_g[\phi] \tag{5.2}$$

coincide with their ADM counterparts[58, 59]. This turns out to be useful in deriving the integral formula without variations, which can provide some hints on what the variational equation should look like.

Another remark worth making is that the Killing vector k^a should Lie-derive both the original metric and the perturbed metric, as a consequence of $\delta k^a = 0$. Then $\mathcal{L}_k \delta g_{ab} = 0$. This can be thought of as a consequence of invariance of the boundary when the variation is performed. And in a sense the vector $(\partial_t)^a$ and $(\partial_\phi)^a$ are defined "from" the boundaries of the leaves in the spacetime foliation. As a consequence, the two integrals in (4.29) do not depend on the choice of hypersurface. To show this, for convenience, denote the integrals as $K(\Sigma)$ and $I(\Sigma)$ respectively

$$K(\Sigma) \equiv \int\limits_{\Sigma} k^a \epsilon_{abcd} T^{ef} \delta g_{ef}, \qquad I(\Sigma) \equiv \int\limits_{\Sigma} k^e T^a{}_e\, \epsilon_{abcd}. \tag{5.3}$$

Let n^a be the future-pointing unit normal of Σ, and let Σ_1 and Σ_2 be two such spacelike hypersurfaces. The convention can be fixed that n_2^a and $-n_1^a$ defines the positively oriented 4-dimensional volume enclosed by these two hypersurfaces, as is shown in Fig 2a.

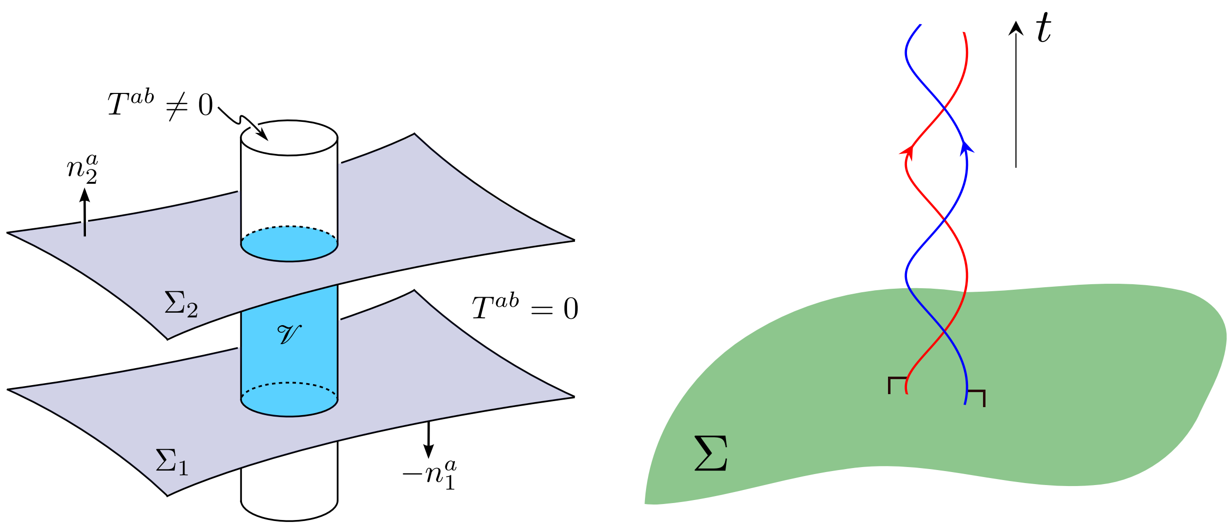
For the integral K,

$$K(\Sigma_2) - K(\Sigma_1) = \int\limits_{\Sigma_1} n_a k^a T^{bc} \delta g_{bc} \hat{\boldsymbol{\epsilon}} - \int\limits_{\Sigma_2} n_a k^a T^{bc} \delta g_{bc} \hat{\boldsymbol{\epsilon}}. \tag{5.4}$$

As is remarked in appendix C.2, the 4-volume form is $\boldsymbol{\epsilon} = -\boldsymbol{n} \wedge \hat{\boldsymbol{\epsilon}}$ for a timelike unit normal, with $\hat{\boldsymbol{\epsilon}}$ being the induced volume form on Σ. Notice that T^{ab} vanishes outside the timelike boundary of the worldtube, hence the difference of $K(\Sigma_2)$ and $K(\Sigma_1)$ gives

$$K(\Sigma_2) - K(\Sigma_1) = -\int\limits_{\partial\mathscr{V}} n_a \left(k^a T^{bc} \delta g_{bc}\right) \hat{\boldsymbol{\epsilon}} = -\int\limits_{\mathscr{V}} \nabla_a \left(k^a T^{bc} \delta g_{bc}\right) \boldsymbol{\epsilon} = 0. \tag{5.5}$$

The second equality makes use of Stokes theorem (see (C.15b)). And the third equality giving 0 makes use of $\nabla_a k^a = 0$ and $k^a \nabla_a (T^{bc} \delta g_{bc}) = \mathcal{L}_k (T^{ab} \delta g_{ab})$, where Leibniz rule for the latter shows that it also vanishes.



(a) Hypersurface independence for I and K

(b) Choice of hypersurface

Figure 2: (a) is used to show the independence of spacelike hypersurface Σ for the integral I and K defined in (5.3), for a matter source T^{ab} with compact support in any spacelike hypersurface. n^a being the future-pointing unit normal vector, the choice of n_2^a and $-n_1^a$ of the worldtube bounds a 4-dimentional region $\mathscr{V}$ where T^{ab} vanishes at the timelike boundary (or "far enough away"). (b) is used to demonstrate the choice of hypersurface Σ for the integration such that $n^a \overset{\mathscr{P}}{=} u^a$ for $\mathscr{P} = \Sigma \cap \gamma$. The "$z$-axis" in Σ is supressed. The perpendicular sign is only schematic for visualization since Lorentzian geometry is different from the Euclidean one.

For the integral $I(\Sigma)$, the calculation is similar.

$$I(\Sigma_2) - I(\Sigma_1) = \int\limits_{\Sigma_1} n_a k_b T^{ab} \hat{\boldsymbol{\epsilon}} - \int\limits_{\Sigma_2} n_a k_b T^{ab} \hat{\boldsymbol{\epsilon}} = -\int\limits_{\partial\mathscr{V}} n_a (k_b T^{ab}) \hat{\boldsymbol{\epsilon}} = -\int\limits_{\mathscr{V}} \nabla_a (k_b T^{ab}) \boldsymbol{\epsilon} = 0. \tag{5.6}$$

The last equality follows again from Killing equation and local energy-momentum conservation.

The freedom to choose spacelike hypersurfaces argued above allows to make the convenient choice, such that, the unit normal coincides with the 4-velocity at the place of the particle, i.e., $\mathscr{P} = \Sigma \cap \gamma$, $n^a \overset{\mathscr{P}}{=} u^a$. A schematic picture is depicted in Fig 2b. This also means that the HKV is normal to Σ at the worldlines. However, it's not the case else where, otherwise k^a will be hypersurface orthogonal and no twist is allowed for such a spacetime, in contradiction to the helical nature.

For such hypersurfaces, the extrinsic curvature K_{ab} can be shown to vanish on the worldlines.

$$K_{ab} \equiv -\frac{1}{2}\mathcal{L}_n \gamma_{ab} \overset{\gamma}{=} -\frac{z^{-1}}{2}\mathcal{L}_k(g_{ab} + u_a u_b) \overset{\gamma}{=} -\frac{1}{z} n_{(a}\mathcal{L}_k n_{b)}. \tag{5.7}$$

$$\begin{aligned} 2n_{(a}\mathcal{L}_k n_{b)} &= n_a\left(k^c\nabla_c n_b + n_c \nabla_b k^c\right) + n_b\left(k^c\nabla_c n_a + n_c\nabla_a k^c\right) \\ &\overset{\gamma}{=} z n_a \dot{n}_b - n_a (z u_b)^{\cdot} + z n_b \dot{u}_a - n_b (z u_a)^{\cdot} \overset{\gamma}{=} 0, \end{aligned} \tag{5.8}$$

where Killing equation for k_a, $n^a \overset{\mathscr{P}}{=} u^a$ and the result $k^a\big|_\gamma = z u^a$ $(\dot{z} = 0)$ mentioned in section 3 are used. Then in the (3+1) decomposition, it can be shown that[60]

$$\nabla_a n_b = -K_{ab} - n_a n^c \nabla_c n_b \overset{\gamma}{=} -u_a \dot{u}_b. \tag{5.9}$$

5.2 Dipolar first law

The current subsection presents the calculation done in [49], which contains the crucial technical tools to extend its result to quadrupole order. Consider the binary system formed by two skeletonized point particle. To recall, at dipolar order, the SEM tensor is given by a sum of (2.28) for the two particles. For simplicity omit the index i and denote $\delta W = K/2 - \delta I$, since the calculation can be done for one particle and a sum is performed in the end. (There's no mention of the trace of extrinsic curvature K_{ab}, so there is no risk of confusion in calling that integral K.)

Evaluate the integral I first.

$$I = -\int_\Sigma n_a k_b T^{ab}\sqrt{h}\mathrm{d}^3x = -\int_\Sigma\int_\gamma n_a k_b u^{(a} p^{b)}\,\delta_4 \mathrm{d}\tau\sqrt{h}\mathrm{d}^3x - \int_\Sigma\int_\gamma n_a k_b \nabla_c\left(u^{(a}S^{b)c}\,\delta_4\right)\mathrm{d}\tau\sqrt{h}\mathrm{d}^3x. \tag{5.10}$$

The (3+1) formalism gives the relation between coordinate time t and proper time τ of an Eulerian observer comoving with the particle in terms of the lapse function N, which satisfies $\mathrm{d}\tau = N\mathrm{d}t$ and $N\sqrt{h} = \sqrt{-g}$. Then

$$\mathrm{d}\tau\sqrt{h}\mathrm{d}^3x = (\mathrm{d}t/N)(N\sqrt{h}\mathrm{d}^3x) = \sqrt{-g}\mathrm{d}^4x \tag{5.11}$$

is a 4-volume element of the spacetime volume $\cup_t \Sigma_t \equiv V$ which is the region of integration. Denote the two integrals in (5.10) as I_1 and I_2 respectively.

$$I_1 = -\int_V n_a k_b u^{(a}p^{b)}\,\delta_4\sqrt{-g}\mathrm{d}^4x = -n_a k_b u^{(a}p^{b)}\big|_\gamma = z u_a p^a = -zm. \tag{5.12}$$

The definition of the rest mass $m \equiv -u_a p^a$ is used. For I_2

$$I_2 = -\int_V \left[\nabla_c\left(n_a k_b u^{(a}S^{b)c}\,\delta_4\right) - u^{(a}S^{b)c}\,\delta_4 \nabla_c(n_a k_b)\right]\sqrt{-g}\mathrm{d}^4x = u^{(a}S^{b)c}\nabla_c(n_a k_b)\big|_\gamma. \tag{5.13}$$

Dropping the boundary term can be justified by Stokes theorem, i.e., the replacement $V \to \partial V$, $\sqrt{-g}\mathrm{d}^4x \to \sqrt{|h|}\mathrm{d}^3y$ and $\nabla_c \to N_c$. ∂V is now a timelike hypersurface and N^a is the unit spacelike outward-pointing normal. The δ_4 restricts the non-vanishing integrand on the worldline, which is enclosed by the boundary ∂V. Hence the boundary term is zero. The remaining expression evaluated on the worldline can be further simplified thanks to (5.9). Define the mass dipole moment $D^b \equiv S^{ab}u_a$, on the worldline,

$$I_2 = \frac{1}{2}\left(D^b\nabla_b(\frac{n^a n_a}{2}) + S^{ab}\nabla_a k_b + zS^{ac}u_c\dot{u}_a - D^c u^b\nabla_b k_c\right) = \frac{1}{2}S^{ab}\nabla_a k_b - D^a\dot{k}_a = \frac{1}{2}S^{ab}\nabla_a k_b + \dot{D}^a k_a. \tag{5.14}$$

The last equality is due to $D^a k_a \stackrel{\gamma}{=} 0$. Finally the result for the integral I is

$$I = -zm + \frac{1}{2}S^{ab}\nabla_a k_b + \dot{D}^a k_a. \tag{5.15}$$

The calculation of the integal K is completely analogous. There's a monopole term which can be directly read out from the invariant Dirac delta, and a dipolar term computed from integration by part. The result is

$$K = k^a p^b \delta g_{ab} + u^b S^{cd}\nabla_d(n_a k^a \delta g_{bc}) = k^a p^b \delta g_{ab} + u^b S^{cd}(0 - \dot{k}_d\delta g_{bc} - z\nabla_d\delta g_{bc}). \tag{5.16}$$

The three terms in the bracket result from (5.9), $u^a\dot{u}_a = 0$ and Killing equation. Using $\delta k^a = 0$, the first term becomes $p^a\delta k_a$. For the second non-vanishing term equivalent to $-S^{cd}\dot{u}_d k^b\delta g_{bc}$,

$$-S^{cd}\dot{u}_d k^b\delta g_{bc} = \left(\dot{D}^c + \dot{S}^{cd}u_d\right)k^b\delta g_{bc} = \left(\dot{D}^c + 2p^{[c}u^{d]}u_d\right)k^b\delta g_{bc} = \dot{D}^a\delta k_a - p^a\delta k_a + mu^a\delta k_a, \tag{5.17}$$

where the definition for the mass dipole moment D^a and rest mass m, as well as the equation of motion for the spin tensor (2.15a) are used. The last term in (5.17) can be further written as

$$mu^a\delta k_a = \frac{m}{z}k^a\delta(g_{ab}k^b)\big|_\gamma = \frac{m}{z}\delta(g_{ab}k^a k^b) = -\frac{m}{z}\delta(z^2) = -2m\delta z \tag{5.18}$$

Eventually the result for K is

$$K = \dot{D}^a\delta k_a - 2m\delta z - k^b S^{cd}\nabla_d\delta g_{bc}. \tag{5.19}$$

Putting everything together, the contribution from one of the particles to the first law of the form $\delta M - \Omega\delta J = \sum_{\mathrm{i}}\delta W_{\mathrm{i}}$, is

$$\begin{aligned}\delta W = \frac{K}{2} - \delta I &= \frac{1}{2}\dot{D}^a\delta k_a - m\delta z - \frac{1}{2}k^b S^{cd}\nabla_d\delta g_{bc} + \delta(zm) - \frac{1}{2}\delta(S^a{}_b\nabla_a k^b) - \delta(\dot{D}^a k_a)\\ &= z\delta m - \delta\dot{D}^a k_a - \frac{1}{2}\dot{D}^a\delta k_a - \frac{1}{2}\nabla_a k^b\delta S^a{}_b - \frac{1}{2}S^{ab}\left[g_{bc}\delta(\nabla_a k^c) - k^c\nabla_a\delta g_{bc}\right].\end{aligned} \tag{5.20}$$

Now the last term with the square bracket can be shown to vanish. Denote the perturbed metric as $\tilde{g}_{ab}$, with the corresponding derivative operator $\tilde{\nabla}_a$ and Christoffel symbols $\tilde{\Gamma}^a_{bc}$. Then

$$g_{bc}\delta(\nabla_a k^c) = g_{bc}\left(\tilde{\nabla}_a \tilde{k}^c - \nabla_a k^c\right) = g_{bc}\delta\Gamma^c_{ad}k^d. \tag{5.21}$$

$$-k^c\nabla_a\delta g_{bc} = -k^c\nabla_a\tilde{g}_{bc} = -k^c(\nabla_a\tilde{g}_{bc} - \tilde{\nabla}_a\tilde{g}_{bc}) = -k^c\delta\Gamma^d_{ac}\, g_{db} - k^c\delta\Gamma^d_{ab}\, g_{cd}, \tag{5.22}$$

where in the last equality $\tilde{g}_{bc}$ is replaced by g_{bc} without inducing higher order correction since $\delta\Gamma$ is already to first order in the perturbation. Combining the two gives $-k^c\delta\Gamma^d_{ab}\, g_{cd}$, a term symmetric of indices ab. Contracting with S^{ab} outside the square bracket will make this term vanish. Eventually the first law of mechanics for dipolar particle binary system, in the most general form, is

$$\delta M - \Omega\delta J = \sum_{\mathsf{i}}\left(z_{\mathsf{i}}\delta m_{\mathsf{i}} - \frac{1}{2}\nabla_a k^b \delta S_{\mathsf{i}}{}^a{}_b - \delta\dot{D}^a_{\mathsf{i}}k_a - \frac{1}{2}\dot{D}^a_{\mathsf{i}}\delta k_a\right). \tag{5.23}$$

To specify the worldline with respect to which a worldline observer measures the spin vector, with three degrees of freedom, three constraints have to be put on the spin tensor S^{ab}, called spin supplementary condition (SSC). A convenient choice is the Frenkel-Mathisson-Pirani SSC $D^a = 0$. The physical intepretation for such an SSC is that the mass dipole moment measured by an observer comoving with the particle vanishes, i.e., the preferred direction, in orther words the dipole moment of the particle, is contributed solely by the spin. In the following whenever an SSC is applied, it refers to this one. Hence, the dipolar first law with SSC is

$$\delta M - \Omega\delta J \overset{\text{SSC}}{=} \sum_{\mathsf{i}}\left(z_{\mathsf{i}}\delta m_{\mathsf{i}} - \frac{1}{2}\nabla_a k^b \delta S_{\mathsf{i}}{}^a{}_b\right). \tag{5.24}$$

A few remarks should be made about the final dipolar result shown here. First, without spin, the dipolar first law reduces to the monopolar first law first established from PN formalism[29]. Once an SSC is specified, a more useful form of the first law will be to write the right hand side in terms of scalars such as spin amplitude. Such a formula, to first order in spin amplitude, has been shown to agree with the result derived from the Hamiltonian formalism[30, 49]. For the quadrupole case, it can be imagined that the right hand side will involve double covariant derivatives, and a coupling of the quadrupole moment with the Riemann tensor. In the attempt to write the quadrupole first law to scalar form, the spin squared term can no longer be ignored, since it will contribute to the spin induced part of the quadrupole first law. Some preliminary results of the first law to quadrupole order are presented in the next subsection.

5.3 Quadrupolar first law: preliminary results

For the quadrupole case, the SEM tensor (2.30) has to be considered instead, which includes two additional terms compared to the dipole case

$$T^{ab}_{\text{quad}} = \frac{1}{3}\int_\gamma R_{cde}{}^{(a}J^{b)cde}\delta_4\,\mathrm{d}\tau - \frac{2}{3}\nabla_{cd}\int_\gamma J^{c(ab)d}\delta_4\,\mathrm{d}\tau. \tag{5.25}$$

The corresponding extra contribution to $\delta W = K/2 - \delta I$ can be computed in the same manner, except the integration by part has to be done twice. The results are

$$K_{\text{quad}} = -\frac{1}{3}n_a k^a R_{def}{}^{b}J^{cdef}\delta g_{bc} + \frac{1}{3}\nabla_{ed}\left(n_a k^a \delta g_{bc}\right)J^{dbce}, \tag{5.26}$$

$$I_{\text{quad}} = -\frac{1}{3}n_a k_b R_{cde}{}^{(a}J^{b)cde} + \frac{2}{3}\nabla_{cd}(n_a k_b)J^{d(ab)c}. \tag{5.27}$$

The reduction of I_{quad} is preferred to be done first since it also enters the integral first law without variations, to be mentioned later. In analogy with (5.9), the double covariant derivative of n^a on the worldline has been calculated in chapter 6 of Ref. [61] to be

$$\nabla_a\nabla_b n_c \overset{\gamma}{=} -\nabla_a K_{bc} + \dot{u}_c(u_a u_b)^{\cdot} + u_a u_b \ddot{u}_c - E_{ac}u_b, \tag{5.28}$$

where E_{ab} is the electric decomposition of Riemann tensor $E_{ab} \equiv R_{acbd}u^c u^d$, whose magnetic conterpart is $B_{ab} \equiv \star R_{acbd}u^c u^d$ with $\star R_{abcd} \equiv (1/2)\epsilon_{ab}{}^{ef}R_{efcd}$. With such a result, Eq. (5.9) and the Kostant formula $\nabla_a\nabla_b k_c = R_{cbad}k^d$ for a Killing vector k^a, the I_{quad} in (5.27) can be further calculated as

$$I_{\text{quad}} = \frac{z}{3}\left(J^{duac}\nabla_c K_{ad} + J^{auub}\dot{u}_a\dot{u}_b + J^{uacu}E_{ca} - R_{cdeu}J^{ucde}\right) + 2J^{ucba}\dot{u}_a\nabla_b k_c + \frac{2z}{3}J^{d(ua)c}R_{adcu}. \tag{5.29}$$

The formulae shown above are the preliminary results for a quadrupolar first law so far. To proceed to an informative first law as a variation formula, several strategies can be adopted to proceed. First, it might be suggestive to first derive an integral formula without any variations. From the expression of Komar mass and angular momentum (5.2), and the specific form of Noether charge 2-form in GR[26], a standard calculation shows [60, 62]

$$\begin{aligned} M_K - 2\Omega J_K &= 2\int_{\partial\Sigma}\boldsymbol{Q}_g[k] = -\frac{1}{8\pi}\int_{\partial\Sigma}\epsilon_{abcd}\nabla^c k^d = \frac{1}{8\pi}\int_{\partial\Sigma}\nabla^c k^d(n_c\wedge N_d)\tilde{\boldsymbol{\epsilon}} \\ &= \frac{1}{4\pi}\int_\Sigma \nabla_d\nabla_c k^d n_c\hat{\boldsymbol{\epsilon}} = \frac{1}{4\pi}\int_\Sigma R_{ab}n^a k^b\hat{\boldsymbol{\epsilon}} = 2\int_\Sigma\left(T^{ab} - \frac{1}{2}Tg^{ab}\right)n_a k_b\sqrt{h}\mathrm{d}^3x, \end{aligned} \tag{5.30}$$

where the first equality of the second line makes use of the second version of Stokes's theorem (see

(C.25) reviewed in appendix C.2.2). $\tilde{\boldsymbol{\epsilon}}$ is the volume form on the 2-sphere $\partial\Sigma$ and $\hat{\boldsymbol{\epsilon}}$ is the volume form on the 3-dimensional spacelike hypersurface as usual. Define the integral

$$L \equiv \int\limits_{\Sigma} T n_a k^a \sqrt{h} \mathrm{d}^3 x. \tag{5.31}$$

It can be shown to be independent of the choice of hypersurface Σ in a similar manner to the argument for I and K in (5.5) and (5.6). Then admitting equality of Komar and ADM quantities, the integral first law can be written conveniently as $M - 2\Omega J = -2I - L$. For example, by inserting the trace of dipolar SEM tensor, the first integral for dipole case is calculated as

$$M - 2\Omega J = \sum_{\mathsf{i}} \left(z_{\mathsf{i}} m_{\mathsf{i}} + D_{\mathsf{i}}^a \dot{k}_a - S_{\mathsf{i}}^{ab} \nabla_a k_b \right) \overset{\text{SSC}}{=} \sum_{\mathsf{i}} \left(z_{\mathsf{i}} m_{\mathsf{i}} - S_{\mathsf{i}}^{ab} \nabla_a k_b \right). \tag{5.32}$$

Although an integral formula contains less information than the variational one, it's still formally reminiscent to (5.23) and (5.24). In fact, once the variational formula is available, a consistency check should be performed for the variational and integral formulae. This is done by writing the formulae in scalar form, and using an argument of Euler theorem for homogeneous function to go from the variational to integral formula. Such a derivation is done for the monopole case [29] and dipole case[49]. The calculation of the integral L for quadrupole case is underway, which will lead to the quadrupole first integral. And once the variational quadrupole first law is available, such a verification is also to be done.

Apart from the extra mathematical complexity, the richer physical content captured by the formulae should also be taken into account. Notice the equation of motion (2.29) does not have an evolution equation for the quadrupole moment J^{abcd}. In fact it has to be specified by physical input. Since it has the algebraic symmetries of Riemann tensor, it can be (3+1)-decomposed as in (B.26) into stress type, momentum type and mass type to match its behavior in classical Newtonian limit. Moreover, it should have a spin-induced part, which accounts for the deformation of an extended spinning body caused by its own spin. The model uses effective field theory and involve a phenomenological polarisability coefficient measuring the tendency of the body to be deformed by its spin. In addition, the body can also be tidally deformed by the tidal field exerted by its companion in the binary system. The tidally induced quadrupole moment will involve the electric type and magnetic type quadrupoles, which describe in the simplest case its adiabatic and linear response to a non-dynamical tidal fields. It will be useful to do the computation of the integrals $K,\ I,\ L$ for the three contributions and see whether they fall in the category of stress, momentum and mass type. Given the physical picture described here, in the final form of the quadrupole first law written in scalar quantities, which is the most desired and useful form, it is anticipated that the additional local contributions to global quantities should involve the coupling between the quadrupole moments and the tidal fields, with potentially the variations of tidal Love numbers as well. Such a result, if obtained, is expected to have implications on modelling the waveforms for binary systems in which one constituent tidally deforms its companion, such as binary neutron stars.

6 Conclusion and outlook

In summary, this work recollects all the prerequisites for extending the first law of mechanics for spinning point particle binary systems in [49] to quadrupole order and derives some preliminary results for this generalization. First, the dynamics of spinning body in curved spacetime which is later used in the derivation of the first law is reviewed in section 2. The equations of motion for the linear momentum and spin tensor are derived via Lagrangian formalism and Tulczyjew reduction in gravitational skeleton formalism. The generalization for quadrupolar particle and the corresponding stress energy momentum (SEM) tensor can be derived analogously but the formalisms will be too involved to be presented in the current text. Results for quadrupolar particle dynamics and SEM tensor are quoted and the interested reader is pointed to [53, 52]. Then picture and motivation of exactly circular orbit for the binary system is brought up in section 3, which reviews the results in Ref. [39]. The properties of the helical Killing vector (HKV) $k^a = (\partial_t)^a + \Omega(\partial_\phi)^a$, to which the circular orbit is translated, are also developed in detail. Notably it has been shown that the two skeletonized point particles' worldlines are actually integral lines of this HKV, and that k^a is related to the 4-velocities of the particles by a constant redshift factor (the so-called Detweiler's redshift observable) along the worldlines. The Lie-dragging of the reduced multipole moments along the worldlines, namely linear momentum p^a, spin tensor S^{ab} and the quadrupole tensor J^{abcd}, is also shown relying on Tulczyjew's skeleton formalism. As a consquence the reduced multipole moments characterizing the particles are geometrically invariant, as should be expected from the physics.

Once the physics of binary system made of skeletonized particles is developed, the mathematical tools to derive the first law for such a system are presented in section 4, reviewing Ref. [25, 26, 56, 57]. In particular, the Noether current and Noether charge associated with a symmetry generated by a vector ξ^a are defined using differential forms, in an arbitrary diffeomorphism covariant theory of gravity involving matter fields in n spacetime dimension. Then a generalization of symplectic mechanics to field theory allows to define and calculate the variation of the conserved quantity generated by ξ^a defined by an integration over a Cauchy surface. The formalism is then applied to the case of stationary axisymmetric spacetime containing a black hole region. It successfully reproduces the first law of black hole thermodynamics $\delta M - \Omega\delta J = (\kappa/2\pi)\delta S$ that was dicovered by Bardeen, Carter and Hawking (BCH) in 1973[24], but without relying too much on the assumptions of the theory of GR as BCH did. Apart from generalizing the black hole thermodynamic first law and giving a local geometrical prescription of black hole entropy, which is the most insightful message of [25], it demonstrates the power and convenience of this method to derive a variational formula relating global quantities to local ones. The differential form method is subsequently applied to GR containing a matter source described by a generic SEM tensor T^{ab} which has compact support on any spacelike hypersurface without interior boundary, in any foliation. The key variational formula (4.1) is derived, which is subsequently applied to the case of skeletonized point particle binary system.

In section 5, before inserting the T^{ab} derived or given in section 2 into the general formula (4.1), a few mathematical properties are established, including the independence of choosing integrating spacelike hypersurfaces, the vanishing of extrinsic curvature on the worldlines and the useful formula from (3+1)-decomposition for a specific convenient choice. Then for the dipole SEM tensor, the two integrals and the subsequent algebraic reductions are done to arrive at the first law (5.23) and (5.24), with and without the Mathisson-Frenkel-Pirani SSC. Some preliminary calculations for the quadrupole case are presented in section 5.3. And some proceeding strategies are also outlined. Specifically, an integral version of the first law can be derived to give a hint on the form of the variational first law. Such a derivation is in preparation. Once the variational first law is available, a consistency check with the integral formula is yet to be performed. Moreover, it is necessary to construct physical models for the quadrupole moment J^{abcd} of the particles since they don't have an evolution equation as shown in section 2. Its three contributions, including the spin-induced, electric and magnetic type, should account for the spin and tidal deformation of the extended body, making use of effective field theory (EFT) and the study of adiabatic and linear tidal response to a non-dynamical tidal field. The ultimate quadrupolar first law, is envisaged to relate $\delta M - \Omega \delta J$ on the left hand side, to scalar expression of variations of quadrupole moments coupled to tidal fields, and possibly the variations of tidal Love numbers. Such a result is still to be derived building on the preliminary results in section 5.3.

Once such a first law is available, on the theory side, it might be interesting to derive this first law by Hamiltonian formalism as cross-checking, similar to [30]. In that direction, the first challenge will be to identify the canonical observables for the phase space. In addition, the spin square contributions should now be taken into account since they contribute to the spin-induced quadrupole moment. The validity and generalization of certain formulae in the dipolar case will demand a further scrutinization. On the other hand, for practical application of the quadrupole first law, the study of tidal deformation, combined with various study of EFT and the equation of state of neutron star, might shed light on calculating the local quantities on the right hand side of the first law. Then the first law might serve as bridge to infer global quantities like the binding energy for such a binary system whose constituents produce tidal fields to deform each other. Such information can turn out to be crucial in inferring the phase of the GW emitted by such a system. Moreover, for realistic purposes, it can also be interesting to the GW community to relax the adiabatic approximation, i.e., to consider the backreaction of emitted GW and the shrinking of the orbit. What modification will be brought to the first law, even for the simpler monopole and dipole case, is still an ongoing effort.

A Tulczyjew's theorems and some explicit normal forms

This appendix presents Tulczyjew's two theorems about multipole expansions in gravitational skeleton formalism[43]. The first theorem is about the existence and uniqueness of multipole expansion into the so-called normal forms. The second provides a sufficient and necessary condition for the vanishing of the tensor expanded in terms of normal forms. For practical reasons the proof is not reviewed, instead explicit expressions for the multipole moments in terms of normal forms are given to quadupole order, which is frequently used in the main text. In the notations used below, capital letters denote a sequence of tensor index. For example $M = a_1 \ldots a_m$. And $\nabla_{i_1 \ldots i_l} = \nabla_{i_1} \ldots \nabla_{i_l}$.

Theorem A.1. *Let $Y^M \equiv Y^{a_1 \ldots a_m}$ be a rank m contravariant tensor field defined only on a timelike worldline γ of a particle in a spacetime $(\mathcal{M}, g_{ab})$, which has multipole expansion*

$$Y^M(x) = \sum_l \nabla_{i_1 \ldots i_l} \int_\gamma \mathcal{Y}^{M i_1 \ldots i_l}(z)\, \delta_4(x, z(\tau))\, \mathrm{d}\tau. \tag{A.1}$$

Then the multipole expansion of the tensor Y^M can be reexpressed as

$$Y^M(x) = \sum_l \nabla_{i_1 \ldots i_l} \int_\gamma y^{M i_1 \ldots i_l}(z)\, \delta_4(x, z(\tau))\, \mathrm{d}\tau, \tag{A.2}$$

where the integrand tensors $y^{M i_1 \ldots i_l}$ satisfy $y^{M i_1 \ldots i_l} = y^{M (i_1 \ldots i_l)}$ and $u_{i_k} y^{M i_1 \ldots i_k \ldots i_l} = 0$ for $k = 1, 2, \ldots, l$, where u^a is the normalized 4-velocity on the worldline. $y^{M i_1 \ldots i_l}$ are called the normal forms and can be expressed in terms of $\mathcal{Y}^{M i_1 \ldots i_l}$. Such an expression, at any given order, always exists and is unique.

Theorem A.2. *Let Y_{i}^M be a set of contravariant rank m tensor, each defined on the worldline γ_{i} of a particle labelled by $\mathsf{i} = \{1, 2, \ldots, p\}$ with $p \in \mathbb{N}^*$. Let $Y^M \equiv \sum_{\mathsf{i}=1}^p Y_{\mathsf{i}}^M$ be the sum of all such rank m tensors. When expanded in normal forms*

$$Y^M(x) = \sum_{\mathsf{i}=1}^p \sum_l \nabla_{i_1 \ldots i_l} \int_{\gamma_{\mathsf{i}}} y_{\mathsf{i}}^{M i_1 \ldots i_l}(z_{\mathsf{i}})\, \delta_4(x, z_{\mathsf{i}}(\tau_{\mathsf{i}}))\, \mathrm{d}\tau_{\mathsf{i}}, \tag{A.3}$$

a sufficient and necessary condition for $Y^M(x)$ to vanish is that all the normal forms in the integrands vanish, i.e., $y_{\mathsf{i}}^{M i_1 \ldots i_l} = 0,\ \forall l \in \mathbb{N}$ and $\forall \mathsf{i} \in \{1, 2, \ldots, p\}$.

Remark. In the literature theorem A.2 is often only shown to hold for a single worldline of a single particle. Appendix C of [39] provides an argument to generalize this result to arbitrary number of particles. Basically the key in the proof of single particle case is to choose a rank m covariant tensor Z_M with compact support $V \subset \mathcal{M}$ that contains the worldline. Contract Z_M with Y^M and use the properties of the normal forms to show that the result vanishes for arbitrary such Z_M if and only if all the normal forms in the integrands vanish. For multi-particle case, one can choose $Z_M^{(\mathrm{i})}$ whose compact support contains the worldline γ_{i} and no other worldlines. Applying the same argument for all the worldlines and the desired result is proven.

Suppose a tensor Y^M is expanded to quadrupole order

$$Y^M(x) = \int\limits_\gamma \mathcal{Y}^M \delta_4 \, \mathrm{d}\tau + \nabla_b \int\limits_\gamma \mathcal{Y}^{Mb} \delta_4 \, \mathrm{d}\tau + \nabla_{cd} \int\limits_\gamma \mathcal{Y}^{Mcd} \delta_4 \, \mathrm{d}\tau, \tag{A.4}$$

with the arguments of tensors understood as in previous formulae. The normal forms can be explicitly calculated. The key is to decompose the integrands into components parallel and orthogonal to the 4-velocity of the particle on the worldline. A useful identity

$$\nabla_a \int\limits_\gamma T^{M'}{}_{K'}(z) u^{a'}(z) \; \delta_4(x, z(\tau)) \, \mathrm{d}\tau = \int\limits_\gamma \dot{T}^{M'}{}_{K'}(z) \; \delta_4(x, z(\tau)) \, \mathrm{d}\tau, \tag{A.5}$$

where $T^M{}_K$ denotes arbitrary (m, k) rank tensor, is frequently used. Unlike before the prime indices are used for the argument z. Property of the invariant Dirac delta $\nabla_a \delta_4(x, z) = -\nabla_{a'} \delta_4(x, z)$, Leibniz rule and omittance of boundary terms are applied to prove this identity. For terms that do not look like the left hand side of (A.5), permuting the covariant derivative can introduce Riemann tensors. Details are provided from (D3) to (D11) in [39]. Only the result is quoted here

$$y^M = \mathcal{Y}^M + \left(\mathcal{Y}^{Mu} - (\mathcal{Y}^{Muu})^{\cdot} + 2\mathcal{Y}^{M(cu)} \dot{u}_c\right)^{\cdot} + \sum_{j=1}^{m} R_{cbe}{}^{a_j} \left(\mathcal{Y}^{M_e u \hat{b}} u^c - \frac{1}{2} \mathcal{Y}^{M_e \hat{c} \hat{b}}\right), \tag{A.6a}$$

$$y^{Mc} = \mathcal{Y}^{M\hat{c}} - 2\left(\mathcal{Y}^{M(du)}\right)^{\cdot} h^c{}_d - \mathcal{Y}^{Muu} \dot{u}^c, \tag{A.6b}$$

$$y^{Mcd} = \mathcal{Y}^{M(\hat{c}\hat{d})}, \tag{A.6c}$$

where $M_e = a_1 \ldots e \ldots a_m$ with a_j replaced by e in the summand. $h^a{}_b = \delta^a{}_b + u^a u_b$ is the projection tensor. Take a rank 3 tensor for example, the u and hat indicies mean, $T^{abu} \equiv T^{abc} u_c$ and $T^{ab\hat{c}} \equiv T^{abd} h^c{}_d$. In words, the u index means the corresponding index is contracted with the 4-velocity u^a and the hat index means the corresponding index is orthogonal to u^a.

B Technicalities of proofs in section 3

B.1 Helical Killing trajectory

First several results about a generic Killing vector field ξ^a should be mentioned and will actually be used along the proof.

$$\mathcal{L}_\xi \nabla_a X = \nabla_a \mathcal{L}_\xi X, \qquad \mathcal{L}_\xi \boldsymbol{\epsilon} = 0, \qquad \mathcal{L}_\xi R_{abcd} = 0, \qquad \mathcal{L}_\xi \delta_4(x,y) = 0, \tag{B.1}$$

where X is an arbitrary tensor field, $\boldsymbol{\epsilon}$ is the volume 4-form, R_{abcd} is the Riemann tensor and $\delta_4(x,y)$ is the invariant Dirac delta distribution. Proofs of these can be found in appendix A.2 of [61].

Let k^a be the HKV under consideration. A consequence of the above results is $\mathcal{L}_k G^{ab} = 0$ and therefore $\mathcal{L}_k T^{ab} = 0$ from Einstein's equation. Expand the SEM tensor to quadrupole order

$$\begin{aligned} T^{ab} &= \sum_{\mathsf{i}} \int_{\gamma_\mathsf{i}} \mathcal{T}_\mathsf{i}^{ab} \delta_4^\mathsf{i} \, \mathrm{d}\tau_\mathsf{i} + \nabla_c \int_{\gamma_\mathsf{i}} \mathcal{T}_\mathsf{i}^{abc} \delta_4^\mathsf{i} \, \mathrm{d}\tau_\mathsf{i} + \nabla_{cd} \int_{\gamma_\mathsf{i}} \mathcal{T}_\mathsf{i}^{abcd} \, \mathrm{d}\tau_\mathsf{i} \\ &= \sum_{\mathsf{i}} \int_{\gamma_\mathsf{i}} \tilde{\mathscr{T}}_\mathsf{i}^{ab} \delta_4^\mathsf{i} \, \mathrm{d}\tau_\mathsf{i} + \nabla_c \int_{\gamma_\mathsf{i}} \tilde{\mathscr{T}}_\mathsf{i}^{abc} \delta_4^\mathsf{i} \, \mathrm{d}\tau_\mathsf{i} + \nabla_{cd} \int_{\gamma_\mathsf{i}} \tilde{\mathscr{T}}_\mathsf{i}^{abcd} \, \mathrm{d}\tau_\mathsf{i}. \end{aligned} \tag{B.2}$$

In the second line the integrands are the multipoles in normal form expressed by $\mathscr{T}_\mathsf{i}$'s in the first line, where the tilde notation stands for the quantities extended by a parallel propagator bitensor constructed from an orthonormal tetrad, so that they are defined in a neighborhood of the worldlines. They can be inserted into the integral thanks to the invariant Dirac delta distributions. With this, one can prove the following lemma:

Lemma B.1. *Let f_{ab} be a covariant rank 2 tensor whose support does not contain the endpoints of the two worldlines $\gamma_{1,2}$. Then*

$$\int_{\gamma_\mathsf{i}} \mathcal{L}_k f_\mathsf{i} \, \mathrm{d}\tau_\mathsf{i} = 0, \text{ where } f_\mathsf{i} \equiv \tilde{\mathscr{T}}_\mathsf{i}^{ab} f_{ab} - \tilde{\mathscr{T}}_\mathsf{i}^{abc} \nabla_c f_{ab} + \tilde{\mathscr{T}}_\mathsf{i}^{abcd} \nabla_{cd} f_{ab}, \tag{B.3}$$

for $\mathsf{i} = 1, 2$ and $\tilde{\mathscr{T}}_\mathsf{i}$'s are the multipole moments in normal form extended by a parallel propagator.

Proof. Let Σ be a closed region in spacetime which contains both worldlines. Then

$$\int_\Sigma \mathcal{L}_k (T^{ab} f_{ab}) \boldsymbol{\epsilon} = \int_\Sigma k^c \nabla_c (T^{ab} f_{ab}) \boldsymbol{\epsilon} = \int_\Sigma \nabla_c (k^c T^{ab} f_{ab}) \boldsymbol{\epsilon} = \int_{\partial\Sigma} n_c k^c T^{ab} f_{ab} \tilde{\boldsymbol{\epsilon}} = 0. \tag{B.4}$$

The second equality comes from $\nabla_c k^c = 0$ implied by Killing equation. The third equality is Stokes's theorem, with $\tilde{\boldsymbol{\epsilon}}$ the volume 3-form induced on the boudary of Σ. The expression vanishes since $f_{ab} T^{ab}$ has compact support in spacetime so that surface terms don't contribute. Since $\mathcal{L}_k T^{ab} = 0$ this also implies $\int_\Sigma T^{ab}\mathcal{L}_k f_{ab}\mathrm{d}V = 0$, where in the context of integration one can always replace $\boldsymbol{\epsilon}$ with $\mathrm{d}V = \sqrt{-g}\,\mathrm{d}^4x$ in a coordinate patch. Substitute (B.2) into the integral, after integration by part and integrating over $\mathrm{d}V$ with the Dirac delta's, while making use of the commutation of Lie-derivatives of Killing field and covariant derivative in (B.1), one arrives at

$$\sum_{\mathsf{i}}\int_{\gamma_{\mathsf{i}}}\left(\tilde{\mathscr{T}}_{\mathsf{i}}^{ab}\mathcal{L}_k f_{ab} - \tilde{\mathscr{T}}_{\mathsf{i}}^{abc}\mathcal{L}_k\nabla_c f_{ab} + \tilde{\mathscr{T}}_{\mathsf{i}}^{abcd}\mathcal{L}_k\nabla_{cd}f_{ab}\right)\mathrm{d}\tau_{\mathsf{i}} = 0, \tag{B.5}$$

which in turn implies

$$\sum_{\mathsf{i}}\int_{\gamma_{\mathsf{i}}}\mathcal{L}_k\left(\tilde{\mathscr{T}}_{\mathsf{i}}^{ab}f_{ab} - \tilde{\mathscr{T}}_{\mathsf{i}}^{abc}\nabla_c f_{ab} + \tilde{\mathscr{T}}_{\mathsf{i}}^{abcd}\nabla_{cd}f_{ab}\right)\mathrm{d}\tau_{\mathsf{i}} = \sum_{\mathsf{i}}\int_{\gamma_{\mathsf{i}}}\left[\mathscr{L}_{\mathsf{i}}^{ab}f_{ab} - \mathscr{L}_{\mathsf{i}}^{abc}\nabla_c f_{ab} + \mathscr{L}_{\mathsf{i}}^{abcd}\nabla_{cd}f_{ab}\right]\mathrm{d}\tau_{\mathsf{i}}, \tag{B.6}$$

where $\mathscr{L}_{\mathsf{i}} \equiv \mathcal{L}_k\tilde{\mathscr{T}}_{\mathsf{i}}$. In the equation $\mathcal{L}_k T^{ab} = 0$ obtained from the second line of (B.2), the $\mathscr{L}$'s possess the index symmetries of the normal form because the $\tilde{\mathscr{T}}$'s being Lie derived are in normal forms, but this does not necessarily imply that $\mathscr{L}$'s have the desired orthogonality to the 4-velocity of the ith particle u_{i}^a for the indices that normal forms should have. Bringing them to normal form and using theorem A.2 puts the following constraints

$$\mathscr{L}_{\mathsf{i}}^{ab} = \left(\mathscr{L}_{\mathsf{i}}^{abu} - (\mathscr{L}_{\mathsf{i}}^{abcu})^{\cdot}u^{\mathsf{i}}_c + \mathscr{L}_{\mathsf{i}}^{abcu}\dot{u}^{\mathsf{i}}_c\right)^{\cdot} - 2R_{cde}{}^{(a}\,\mathscr{L}_{\mathsf{i}}^{b)eu\hat{d}}u_{\mathsf{i}}^c \tag{B.7a}$$
$$\mathscr{L}_{\mathsf{i}}^{abc} = -\mathscr{L}_{\mathsf{i}}^{abu}u_{\mathsf{i}}^c + 2(\mathscr{L}_{\mathsf{i}}^{abdu})^{\cdot}\,h^c_{\mathsf{i}\;d} + \mathscr{L}_{\mathsf{i}}^{abuu}\dot{u}_{\mathsf{i}}^c \tag{B.7b}$$
$$\mathscr{L}_{\mathsf{i}}^{abcd} = -2\mathscr{L}_{\mathsf{i}}^{abu\left(\hat{d}\right.}u_{\mathsf{i}}^{\left.c\right)} + \mathscr{L}_{\mathsf{i}}^{abuu}u_{\mathsf{i}}^c u_{\mathsf{i}}^d. \tag{B.7c}$$

Inserting (B.7) into the right hand side of (B.6), while admitting that the covariant total derivative $\mathrm{D}/\mathrm{d}\tau_{\mathsf{i}} \equiv u_{\mathsf{i}}^a\nabla_a$ can always be integrated by part while ignoring the boundary terms (because each term contains f_{ab}), after all the cancellations (detailed below) one can arrive at a surprisingly simple result

$$\sum_{\mathsf{i}}\int_{\gamma_{\mathsf{i}}}\mathcal{L}_k\left(\tilde{\mathscr{T}}_{\mathsf{i}}^{ab}f_{ab} - \tilde{\mathscr{T}}_{\mathsf{i}}^{abc}\nabla_c f_{ab} + \tilde{\mathscr{T}}_{\mathsf{i}}^{abcd}\nabla_{cd}f_{ab}\right)\mathrm{d}\tau_{\mathsf{i}} = \sum_{\mathsf{i}}\int_{\gamma_{\mathsf{i}}}\left(f_{ab}\mathscr{L}_{\mathsf{i}}^{abu}\right)^{\cdot}\,\mathrm{d}\tau_{\mathsf{i}}. \tag{B.8}$$

This is shown by plugging (B.7) into the right hand side of (B.6). The property $\mathscr{L}^{ab\cdots} = \mathscr{L}^{(ab)\cdots}$ and $\mathscr{L}^{ab\cdots} = \mathscr{L}^{ab(\cdots)}$ can be used since it's the Lie derivative of the normal form unreduced multipole moments derived from a SEM tensor T^{ab}. Work explicitly on the right hand side of (B.6). For the first term

$$\begin{aligned}\mathscr{L}^{ab} f_{ab} &= f_{ab}\left[\mathscr{L}^{abu} - \left(\mathscr{L}^{abcu}\right)^{\cdot} u_c + \mathscr{L}^{abcu}\dot{u}_c\right]^{\cdot} - f_{ab}\left(R_{cde}{}^{a}\mathscr{L}^{beu\hat{d}}u^c + R_{cde}{}^{b}\mathscr{L}^{aeu\hat{d}}u^c\right) \\ &= f_{ab}\left[\left(\mathscr{L}^{abu}\right)^{\cdot} - \left(\mathscr{L}^{abcu}\right)^{\cdot\cdot} u_c - \left(\mathscr{L}^{abcu}\right)^{\cdot}\dot{u}_c + \mathscr{L}^{abcu}\ddot{u}_c + \left(\mathscr{L}^{abcu}\right)^{\cdot}\dot{u}_c \right.\\ &\qquad\qquad \left. -u^c\left(R_{cde}{}^{a}\mathscr{L}^{beu\hat{d}} + R_{cde}{}^{b}\mathscr{L}^{aeu\hat{d}}\right)\right] \\ &= f_{ab}\left[\left(\mathscr{L}^{abu}\right)^{\cdot} - \left(\mathscr{L}^{abcu}\right)^{\cdot\cdot} u_c + \mathscr{L}^{abcu}\ddot{u}_c - u^c\left(R_{cde}{}^{a}\mathscr{L}^{beu\hat{d}} + R_{cde}{}^{b}\mathscr{L}^{aeu\hat{d}}\right)\right].\end{aligned} \tag{B.9}$$

The second term becomes

$$\begin{aligned}-\mathscr{L}^{abc}\nabla_c f_{ab} &= \nabla_c f_{ab}\left[\mathscr{L}^{abu}u^c - 2(\mathscr{L}^{abdu})^{\cdot}(\delta^c{}_d + u^c u_d) - \mathscr{L}^{abuu}\dot{u}^c\right] \\ &= \mathscr{L}^{abu}\dot{f}_{ab} - 2(\mathscr{L}^{abcu})^{\cdot}\nabla_c f_{ab} - 2(\mathscr{L}^{abcu})^{\cdot}u_c\dot{f}_{ab} - \mathscr{L}^{abuu}\dot{u}^c\nabla_c f_{ab}.\end{aligned} \tag{B.10}$$

And for the third term

$$\begin{aligned}\mathscr{L}^{abcd}\nabla_{cd} f &= \left(-\mathscr{L}^{abu\hat{d}}u^c - \mathscr{L}^{abu\hat{c}}u^d\right)\nabla_c\nabla_d f_{ab} + \mathscr{L}^{abuu}u^c u^d\nabla_c\nabla_d f_{ab} \\ &= -\mathscr{L}^{abu\hat{c}}(\nabla_c f_{ab})^{\cdot} - \mathscr{L}^{abu\hat{c}}u^d\nabla_c\nabla_d f_{ab} + \mathscr{L}^{abuu}u^c(\nabla_c f_{ab})^{\cdot} \\ &= -\mathscr{L}^{abuc}(\nabla_c f_{ab})^{\cdot} - \mathscr{L}^{abu\hat{c}}u^d(\nabla_c\nabla_d - \nabla_d\nabla_c + \nabla_d\nabla_c)f_{ab} \\ &= -\mathscr{L}^{abuc}(\nabla_c f_{ab})^{\cdot} - -\mathscr{L}^{abu\hat{c}}(\nabla_c f_{ab})^{\cdot} - \mathscr{L}^{abu\hat{c}}u^d\left(R_{cda}{}^{e}f_{eb} + R_{cdb}{}^{e}f_{ae}\right),\end{aligned} \tag{B.11}$$

where in the third line the decomposition $\mathscr{L}^{abuc} = \mathscr{L}^{abu\hat{c}} - \mathscr{L}^{abuu}u^c$ is used. The first terms for the final results of (B.9) and (B.10) combines into $(\mathscr{L}^{abu}f_{ab})^{\cdot}$. A careful examination of the terms involving the Riemann tensors in (B.9) and (B.11) gives

$$\begin{aligned}\text{Riemann terms} &= -f_{ab}u^c R_{cde}{}^{a}\mathscr{L}^{beu\hat{d}} - f_{ab}u^c R_{cde}{}^{b}\mathscr{L}^{aeu\hat{d}} - \mathscr{L}^{abu\hat{c}}u^d R_{cda}{}^{e}f_{eb} - \mathscr{L}^{abu\hat{c}}u^d R_{cdb}{}^{e}f_{ae} \\ &= -2R_{cde}{}^{a}f_{ab}\mathscr{L}^{beu(\hat{c}}u^{d)} - 2R_{cde}{}^{b}f_{ab}\mathscr{L}^{aeu(\hat{c}}u^{d)} \\ &= 0.\end{aligned} \tag{B.12}$$

The remaining terms all contain $\mathscr{L}$ with "4 indices" (in which contraction with u also counts). They are

$$\begin{aligned}\text{remaining terms} &= -f_{ab}(\mathscr{L}^{abcu})^{\cdot\cdot}u_c + f_{ab}\mathscr{L}^{abcu}\ddot{u}_c - 2(\mathscr{L}^{abcu})^{\cdot}\nabla_c f_{ab} - 2(\mathscr{L}^{abcu})u_c\dot{f}_{ab} \\ &\quad - \mathscr{L}^{abuu}\dot{u}^c\nabla_c f_{ab} - \mathscr{L}^{abuc}\left(\nabla_c f_{ab}\right)^{\cdot} - \mathscr{L}^{abu\hat{c}}(\nabla_c f_{ab})^{\cdot}.\end{aligned} \tag{B.13}$$

The terms are then divided into two groups and calculated separately, based on the presence of $\nabla_c f_{ab}$. Notice that each term contains f_{ab} whose support excludes the endpoint of the worldline γ, in the context of integration on the right hand side of (B.6), an integration by part can always be effectuated.

In the following, $\simeq$ means equal up to integration by part.

$$\begin{aligned}
\text{group 1} &= -f_{ab}(\mathscr{L}^{abcu})^{\cdot\cdot}u_c + f_{ab}\mathscr{L}^{abcu}\ddot{u}_c - 2(\mathscr{L}^{abcu})u_c\dot{f}_{ab} \\
&\simeq -f_{ab}(\mathscr{L}^{abcu})^{\cdot\cdot}u_c + u_c\left(f_{ab}\mathscr{L}^{abcu}\right)^{\cdot\cdot} - 2u_c\dot{f}_{ab}(\mathscr{L}^{abcu})^{\cdot} \\
&= -f_{ab}(\mathscr{L}^{abcu})^{\cdot\cdot}u_c + u_c\left[(\mathscr{L}^{abcu})^{\cdot\cdot}f_{ab} + 2\dot{f}_{ab}(\mathscr{L}^{abcu})^{\cdot} + \mathscr{L}^{abcu}\ddot{f}_{ab}\right] - 2u_c(\mathscr{L}^{abcu})^{\cdot}\dot{f}_{ab} \\
&= \mathscr{L}^{abcu}u_c\ddot{f}_{ab} = \mathscr{L}^{abuu}\ddot{f}_{ab}.
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\text{group 2} &= -2(\mathscr{L}^{abcu})^{\cdot}\nabla_c f_{ab} - \mathscr{L}^{abuu}\dot{u}^c\nabla_c f_{ab} - \mathscr{L}^{abuc}(\nabla_c f_{ab})^{\cdot} - \mathscr{L}^{abu\hat{c}}(\nabla_c f_{ab})^{\cdot} \\
&\simeq 2\mathscr{L}^{abcu}(\nabla_c f_{ab})^{\cdot} + u^c(\mathscr{L}^{abuu}\nabla_c f_{ab})^{\cdot} - \mathscr{L}^{abuc}(\nabla_c f_{ab})^{\cdot} - \mathscr{L}^{abu\hat{c}}(\nabla_c f_{ab})^{\cdot} \\
&= (\mathscr{L}^{abcu} - \mathscr{L}^{ab\hat{c}u})(\nabla_c f_{ab})^{\cdot} + u^c(\mathscr{L}^{abuu})^{\cdot}\nabla_c f_{ab} + u^c\mathscr{L}^{abuu}(\nabla_c f_{ab})^{\cdot} \\
&= u^c(\mathscr{L}^{abuu})^{\cdot}\nabla_c f_{ab} = (\mathscr{L}^{abuu})^{\cdot}(\nabla_c f_{ab})^{\cdot} \\
&\simeq -\mathscr{L}^{abuu}\ddot{f}_{ab},
\end{aligned} \tag{B.15}$$

where the decomposition $\mathscr{L}^{abcu} = \mathscr{L}^{ab\hat{c}u} - \mathscr{L}^{abuu}u^c$ is again used to arrive at the fourth line. Surprising all the terms cancel out except the first terms of (B.9) and (B.10) yielding $(\mathscr{L}^{abu}f_{ab})^{\cdot}$. Hence (B.8) is shown.

The integrals for $\mathsf{i} = 1, 2$ on the left hand side of (B.8) are just in the form of the integral appearing in (B.3), and each one in the sum vanishes since it is a total derivative and the support of f_{ab} excludes the endpoints of both worldlines. The lemma is thus proven. □

Once lemma B.1 is true, the following proposition can be proven

Proposition B.1. *Lemma B.1 is a sufficient and necessary condition for the following statement: the helical Killing field k^a evaluated on the worldlines of the two particles is proportional to their 4-velocities, with constant proportionality constant along each worldline, i.e., $k^a\big|_{\gamma_\mathsf{i}} = z_\mathsf{i} u_\mathsf{i}^a$ with $\dot{z}_\mathsf{i}(y(\tau_\mathsf{i})) = 0,\ \forall y \in \gamma_\mathsf{i}$.*

Proof. The necessary direction is easy. If $k^a\big|_{\gamma_\mathsf{i}} = z_\mathsf{i} u_\mathsf{i}^a$ with $\dot{z}_\mathsf{i}(y(\tau_\mathsf{i})) = 0$ on the worldlines, then $\int_{\gamma_\mathsf{i}} \mathcal{L}_k f_\mathsf{i}\, \mathrm{d}\tau_\mathsf{i} \propto \int_{\gamma_\mathsf{i}} \dot{f}_\mathsf{i} \mathrm{d}\tau_\mathsf{i} = 0\ \forall f_\mathsf{i} \in \mathbb{F} := \left\{ \tilde{\mathscr{T}}_\mathsf{i}^{ab} f_{ab} - \tilde{\mathscr{T}}_\mathsf{i}^{abc}\nabla_c f_{ab} + \tilde{\mathscr{T}}_\mathsf{i}^{abcd}\nabla_{cd} f_{ab} | \partial\gamma_{1,2} \notin \operatorname{supp}(f_{ab}) \right\}$.

The proof of the sufficient direction exploits contraposition. Decompose $k^a = zu^a + w^a$, where $z = -u^a k_a$ and $w^a = h^a{}_b k^b = (\delta^a{}_b + u^a u_b)k^b$. Construct Fermi normal coordinate in a neighborhood of the worldline (t, x^1, x^2, x^3), with $t = \tau$ the proper time of the particle. Specifically the Christoffel symbols vanish on the worldline $\Gamma^\rho_{\mu\nu}\big|_\gamma = 0$, and the vectors have components $u^\mu = (1, 0, 0, 0)$, $w^\mu = (0, w^1, w^2, w^3)$.

Suppose $\exists\, t$ such that $\dot{z}(t) \neq 0$, then by continuity there exists a neighborhood of t, say (τ_-, τ_+), such that $\dot{z}(t) \neq 0$ and is of constant sign for $t \in (\tau_-, \tau_+)$. Choose a function expressed in Fermi normal coordinate as

$$f(t,x^i)=\begin{cases}\exp\left[\frac{1}{(t-\tau_-)(t-\tau_+)}\right] & \text{for } t\in(\tau_-,\tau_+),\\ 0 & \text{otherwise.}\end{cases} \tag{B.16}$$

This function satisfies $\int_\gamma \mathcal{L}_k f\,\mathrm{d}\tau=\int_\gamma(zu^a\nabla_a f+w^a\nabla_a f)\,\mathrm{d}\tau=\int_\gamma z\dot{f}\,\mathrm{d}\tau=-\int_{\tau_-}^{\tau_+}\dot{z}f\,\mathrm{d}\tau\neq 0$. If there exists a covariant rank 2 tensor f_{ab} with the property $\partial\gamma\notin\mathrm{supp}(f_{ab})$ whose image $\tilde{f}\in C^\infty(\mathcal{M},\mathbb{R})$ under the map $\mathscr{F}:T^{(0,2)}\mathcal{M}\to C^\infty(\mathcal{M},\mathbb{R})$ defined by

$$\tilde{f}\equiv\mathscr{F}(f_{ab}):=\tilde{\mathscr{T}}^{ab}f_{ab}-\tilde{\mathscr{T}}^{abc}\nabla_c f_{ab}+\tilde{\mathscr{T}}^{abcd}\nabla_{cd}f_{ab}, \tag{B.17}$$

where $\tilde{\mathscr{T}}$'s are the parallel propogated normal form multipoles of T^{ab} as appeared in lemma B.1, generates the same result as f in (B.16) when fed in the worldline integral, i.e., $\int_\gamma\mathcal{L}_k\tilde{f}\,\mathrm{d}\tau=\int_\gamma\mathcal{L}_k f\,\mathrm{d}\tau\neq 0$. Lemma B.1 will be contradicted since by construction $\tilde{f}\in\mathbb{F}$. An ansatz of f_{ab} is $\phi(x)g_{ab}$. Then (B.17) in Fermi normal coordinate evaluated on the worldline reads

$$\tilde{f}(t,x^i)=\tilde{\mathscr{T}}^{\mu\nu}g_{\mu\nu}\phi-\tilde{\mathscr{T}}^{\mu\nu i}g_{\mu\nu}\partial_i\phi+\tilde{\mathscr{T}}^{\mu\nu ij}g_{\mu\nu}\partial_i\partial_j\phi. \tag{B.18}$$

$\tilde{\mathscr{T}}^{\mu\nu 0}$, $\tilde{\mathscr{T}}^{\mu\nu\rho 0}$, $\tilde{\mathscr{T}}^{\mu\nu 0\rho}$ vanish because they are in normal form and $u^\mu=(1,0,0,0)$. This equation only holds on the worldline since the Christoffel symbols are ignored. And all the functional arguments are understood as $(t,\vec{0})$. Clearly, the choice of $\phi(t,x^i)=\left(\tilde{\mathscr{T}}^{\mu\nu}(t,\vec{0})g_{\mu\nu}(t,\vec{0})\right)^{-1}f$ fulfils the requirement, provided that the f appearing here is the one in (B.16). Hence we have found an $\tilde{f}\in\mathbb{F}$ to violate lemma B.1. But since the lemma has been proven true, this argument implies that $\dot{z}$ must vanish throughout the worldline.

Now, suppose $\exists\ t$ such that $w^a(t)\neq 0$. By continuity there exists a neighborhood of t, say (τ_-,τ_+), such that $w^a\neq 0$. In Fermi normal coordinate, without loss of generality, suppose $w^\mu=(0,w^1,0,0)$ and the continuity condition guarantees that in a small enough neighborhood (τ_-,τ_+) of t, w^1 is of constant sign. The Lie-derived worldline integral in component for a function f is $\int_\gamma\mathcal{L}_k f\,\mathrm{d}\tau=\int_\gamma(-\dot{z}f+w^i\partial_i f)\,\mathrm{d}\tau=\int_\gamma w^i\partial_i f\,\mathrm{d}\tau$ since $\dot{z}$ is proven to vanish. In a similar spirit the function f that makes this integral non-zero can be chosen as

$$f(t,x^i)=\begin{cases}x^1\exp\left[\frac{1}{(t-\tau_-)(t-\tau_+)}\right] & \text{for } t\in(\tau_-,\tau_+),\\ 0 & \text{otherwise.}\end{cases} \tag{B.19}$$

Again, the goal is to find an f_{ab} with the condition $\partial\gamma\notin\mathrm{supp}(f_{ab})$ such that $\tilde{f}=\mathscr{F}(f_{ab})$ satisfies $\int_\gamma\mathcal{L}_k\tilde{f}\,\mathrm{d}\tau=\int_\gamma\mathcal{L}_k f\,\mathrm{d}\tau\neq 0$. Consider the ansatz $f_{ab}=\phi g_{ab}$. On the worldline

$$\tilde{f}(t,x^1)=\tilde{\mathscr{T}}^{\mu\nu}g_{\mu\nu}\phi-\tilde{\mathscr{T}}^{\mu\nu 1}g_{\mu\nu}\partial_1\phi+\tilde{\mathscr{T}}^{\mu\nu 11}g_{\mu\nu}\partial_1^2\phi. \tag{B.20}$$

Again, functional arguments as $(t, \vec{0})$ are understood in this equation. Unlike the case for proving $\dot{z} = 0$, this time one wants $\partial_1 \tilde{f}$ and $\partial_1 f$ to agree on the worldline since the Lie derivative contributing non-trivially to the worldline integral reads in component $\mathcal{L}_k f = w^1 \partial_1 f$. Denote the function in (B.16) as $l(t)$ to avoid confusion. For taking ∂_1 on both sides of (B.20), analogously the choice of $\phi(t, x^1)$ can be

$$\phi(t, x^1) = x^1 l(t) \left(\tilde{\mathscr{T}}^{\mu\nu}(t, \vec{0}) g_{\mu\nu}(t, \vec{0}) \right)^{-1}. \tag{B.21}$$

Then $f_{ab} = \phi g_{ab}$ is the desired tensor satisfying $\partial\gamma \notin \operatorname{supp}(f_{ab})$ because of the $l(t)$ in its construction, for which $\tilde{f} = \mathscr{F}(f_{ab})$ satisfies

$$\int_\gamma \mathcal{L}_k \tilde{f} \, \mathrm{d}\tau = \int_\gamma w^i \partial_i \tilde{f} \, \mathrm{d}\tau = \int_{\tau_-}^{\tau_+} w^1 l(t) \, \mathrm{d}\tau = \int_\gamma \mathcal{L}_k f \, \mathrm{d}\tau \neq 0 \tag{B.22}$$

The upshot, is that if w^a does not vanish throughout the worldline, lemma B.1 is also violated by choosing a function $\tilde{f} \in \mathbb{F}$ such that $\int_\gamma \mathcal{L}_k \tilde{f} \, \mathrm{d}\tau \neq 0$. By contraposition, w^a vanishes throughout the worldline. With the proof of $\dot{z}$ vanishes along the worldline, this basically concludes the proof of sufficient direction of proposition B.1.

A potential flaw is to have two time intervals on the worldline such that $\dot{z}$ or w^a flips sign, the same choice of counter example scalar function f will still make the Lie-derived worldline integral (B.3) vanish. However, in the physical picture of point particle binary system, z is the Detweiler's redshift observable, i.e., the amount of redshift observed by a receiver sitting at the symmetry axis of the circular orbit for light emitted from one of the particles. The shrinking of orbit in reality forbids $\dot{z}$ to have different signs if it is ever *allowed* to be non-zero, albeit the shrinking does not occur in the adiabatic approximation in the main text. In addition, since w^1 in the argument above is the spacelike component of a helical vector $k^a = (\partial_t)^a + \Omega(\partial_\phi)^a$. If it's ever *allowed* to be non-zero, the continuity of physical process will also exclude the case where w^1 flips sign in two intevals since one cannot have $k^a = (\partial_t)^a + \Omega(\partial_\phi)^a$ switching to $k^a = (\partial_t)^a - \Omega(\partial_\phi)^a$ at some point. These requirements from physics rule out this potential caveat, hence the proposition is proven. □

Remark. (i) The authors of Ref. [39] try to argue that the counter examples called f in (B.16) and (B.19) *are* actually in $\mathbb{F}$. However a more appropriate argument presented here, is that one only needs to find a tensor f_{ab} with property $\partial\gamma \notin \operatorname{supp}(f_{ab})$ such that $\tilde{f} = \mathscr{F}(f_{ab})$ as given by (B.17) *produces the same result as the counter example f when fed into the worldline integral of the Lie-derived function*! The reason is that computations in Fermi normal coordinate can only be done on the worldline, e.g., the key equation (B.18) when an ansatz $f_{ab} = \phi g_{ab}$ is taken. One does not have control over the the Christoffel symbols because the metric is simply taken as given. In the case of proving $\dot{z} = 0$, it happens to be the case that $\tilde{f}$ and f agree on the worldline. But because f in (B.19) is 0 evaluated at $x^i = 0$, i.e., the worldline, the same attempt will yield a trivial identity $0 = 0$ on the worldline. Instead, to argue the violation of lemma B.1 by contraposition, one has to ensure $\partial_1 \tilde{f} = \partial_1 f$. This makes $\tilde{f}$ different from f, but it's okay since they produce the same effect when Lie-derived and then integrated along the worldline, i.e., $\int_\gamma \mathcal{L}_k \tilde{f}\,\mathrm{d}\tau = \int_\gamma \mathcal{L}_k f\,\mathrm{d}\tau \neq 0$. And by construction $\tilde{f} \in \mathbb{F}$ is the desired counter example.

(ii) The additional caveat mentioned at the end of the proof and its remedy by physical requirement is also a point not discussed before. It means that the proposition, at least if argued in the thread presented here, cannot be proven solely from mathematics. The result that the point particles in the binary follow the integral curves of the HKV is a natural result that should be expected from the physical picture. However, the original proof in [39] does not rely on the helical nature and seems applicable to any Killing field. The fixing of this potential flaw made use of the helical nature, which is reassuring in some sense.

B.2 Lie-dragging of reduced multipoles

This subsection should prove the following statement:

Proposition B.2. *The reduced multipole moments for a quadrupolar particle in gravitational skeleton formalism, i.e., the linear momentum p^a, spin tensor S^{ab} and quadrupole tensor J^{abcd} as obtained in (2.30) are all Lie-dragged along the particle's worldline, meaning*

$$\mathcal{L}_u p^a = 0, \qquad \mathcal{L}_u S^{ab}, \qquad \mathcal{L}_u J^{abcd} = 0, \tag{B.23}$$

where u^a is the normalized 4-velocity of the particle on the worldline.

Proof. It is proven in proposition B.1 that u^a is proportional to the HKV k^a on the worldline with fixed proportionality constant, it suffices to prove the case with u^a replaced by k^a. From (2.30), $\mathcal{L}_k T^{ab} = 0$ gives

$$\int\limits_\gamma \left(u^{(a}\mathcal{P}^{b)} + \frac{1}{3} R_{cde}{}^{(a}\mathcal{J}^{b)cde} \right) \delta_4\,\mathrm{d}\tau + \nabla_c \int\limits_\gamma u^{(a}\mathcal{S}^{b)c}\delta_4\,\mathrm{d}\tau + \nabla_{cd} \int\limits_\gamma \left(-\frac{2}{3}\mathcal{J}^{c(ab)d} \right) \delta_4\,\mathrm{d}\tau = 0, \tag{B.24}$$

where $\mathcal{P}^a \equiv \mathcal{L}_k p^a$, $\mathcal{S}^{ab} \equiv \mathcal{L}_k S^{ab}$ and $\mathcal{J}^{abcd} \equiv \mathcal{L}_k J^{abcd}$. Results in (B.1) are used, as well as $\mathcal{L}_k u^a = 0$ since the two vectors are colinear. The next step is to bring (B.24) to normal forms and see how their vanishing due to Tulczjew's second theorem A.2 puts constraints on the Lie derived reduced multipole moments. Identifying from the three integrals the $\mathcal{Y}^{ab}$, $\mathcal{Y}^{abc}$, $\mathcal{Y}^{abcd}$ in (A.6) will give the vanishing normal forms expressed in terms of the u, $\mathcal{P}$, $\mathcal{S}$ and $\mathcal{J}$. The latter is very simple and reads

$$y^{abcd} = \mathcal{Y}^{ab(\hat{c}\hat{d})} = -\frac{2}{3}\mathcal{J}^{(\hat{c}|(ab)|\hat{d})} = 0. \tag{B.25}$$

Use of algebraic symmetries of the Riemann tensor which $\mathcal{J}^{abcd}$ shares will give $\mathcal{J}^{\hat{c}(ab)\hat{d}} = 0$. Furthermore, the 4-velocity decomposition of $\mathcal{J}^{abcd}$ is given in [39] as

$$\mathcal{J}^{abcd} = \hat{\mathcal{J}}^{abcd} + 2u^{[a}\mathcal{J}^{b]cd} + 2u^{[d}\mathcal{J}^{c]ba} - 4u^{[a}\mathcal{J}^{b][c}u^{d]}, \tag{B.26}$$

where the various $\mathcal{J}$ on the right hand side are defined as

$$\hat{\mathcal{J}}^{abcd} \equiv \mathcal{J}^{\hat{a}\hat{b}\hat{c}\hat{d}}, \qquad \mathcal{J}^{abc} \equiv \mathcal{J}^{\hat{a}u\hat{b}\hat{c}} \qquad \mathcal{J}^{ab} = \mathcal{J}^{\hat{a}u\hat{b}u}. \tag{B.27}$$

Symmetrize bc indices of $\mathcal{J}^{ebcf}$ expressed in (B.26) and contract both sides with $h^a{}_e h^d{}_f$. This gives, along with $\mathcal{J}^{\hat{c}(ab)\hat{d}} = 0$ and $\mathcal{J}^{abc} = -\mathcal{J}^{acb}$

$$\hat{\mathcal{J}}^{a(bc)d} + \mathcal{J}^{(ad)(b}u^{c)} - u^b u^c \mathcal{J}^{ad} = 0. \tag{B.28}$$

Contracting both sides with $u_b u_c$ yields $\mathcal{J}^{ad} = 0$. Then the same equation free of the third term contracted with u_c will give $\mathcal{J}^{adb} + \mathcal{J}^{dab} = 0$, i.e., $\mathcal{J}^{abc} = \mathcal{J}^{[ab]c}$. Together with $\mathcal{J}^{abc} = \mathcal{J}^{a[bc]}$, the cyclic permutation sum reads

$$\mathcal{J}^{abc} + \mathcal{J}^{bca} + \mathcal{J}^{cab} = 3\mathcal{J}^{abc} = \mathcal{J}^{\hat{a}u\hat{b}\hat{c}} + \mathcal{J}^{\hat{b}u\hat{c}\hat{a}} + \mathcal{J}^{\hat{c}u\hat{a}\hat{b}} = \mathcal{J}^{\hat{b}\hat{c}\hat{a}u} + \mathcal{J}^{\hat{c}\hat{a}\hat{b}u} + \mathcal{J}^{\hat{a}\hat{b}\hat{c}u} = 0. \tag{B.29}$$

Hence $\mathcal{J}^{abc} = 0$. Then (B.26) implies that now $\mathcal{J}^{abcd} = \hat{\mathcal{J}}^{abcd}$. With the property $\mathcal{J}^{a(bc)d} = 0$, i.e., $\mathcal{J}^{abcd} = \mathcal{J}^{a[bc]d}$ shown before, and the algebraic symmetry of Riemann tensor, it's trivial to show that $\mathcal{J}^{abcd} = 0$. Hence we've shown that the normal form constraint (B.25) and the decomposition (B.26) forces $\mathcal{L}_k J^{abcd} = \mathcal{J}^{abcd}$ to vanish identically.

With this, the remaining two normal forms of (B.24) are greatly simplified because the 4-index terms in the general formula (A.6) all drop out. The result is

$$y^{ab} = u^{(a}\mathcal{P}^{b)} - \left(u^{(a}\mathcal{S}^{b)u}\right)^{\cdot} = 0, \tag{B.30a}$$

$$y^{abc} = u^{(a}\mathcal{S}^{b)\hat{c}} = 0. \tag{B.30b}$$

Contract (B.30b) with u_a, and use $\mathcal{S}^{b\hat{c}} = \mathcal{S}^{\hat{b}\hat{c}} - u^b\mathcal{S}^{u\hat{c}}$. It gives $\mathcal{S}^{\hat{b}\hat{c}} = 0$. Then (B.30b) free of $\mathcal{S}^{\hat{b}\hat{c}}$ becomes $-u^a u^b \mathcal{S}^{u\hat{c}} = 0$. Contracting both sides with $u_a u_b$ gives $\mathcal{S}^{u\hat{c}} = 0$. Then the 4-velocity decomposition of the anti-symmetric $\mathcal{S}^{ab}$ simply gives $\mathcal{S}^{ab} = \mathcal{S}^{\hat{a}\hat{b}} = 0$. With this, the same contraction trick for $u^{(a}\mathcal{P}^{b)} = 0$ implied by (B.30a) gives $\mathcal{P}^a = 0$. Hence it is proven that

$$\mathcal{J}^{abcd} = \mathcal{L}_k J^{abcd} = 0, \qquad \mathcal{S}^{ab} = \mathcal{L}_k S^{ab} = 0, \qquad \mathcal{P}^a = \mathcal{L}_k p^a = 0. \tag{B.31}$$

□

C A review of Noether and Stokes theorem

This appendix provides a collection of pedagogical reviews about Noether theorem, Stokes theorem and differential forms. A correspondence of abstract and component representation will be provided for the latter.

C.1 Noether theorem in flat spacetime

The purpose of giving the Noether theorem in such a framework is to compare with that in curved spacetime and derived using differential forms, in order to get some heuristics.

The statement of Noether theorem is that for each global symmetry there corresponds a conserved 4-current when the field equation of motion is satisfied. Consider a theory for a collection of fields $\{\phi_i\}$ with support M in Minkowski spacetime, whose dynamics is given by an action

$$S[\phi_i] = \int_M \mathrm{d}^4x\, \mathcal{L}(\phi_i, \partial\phi_i). \tag{C.1}$$

Consider an infinitesimal transformation

$$\begin{cases} x^\mu \to x'^\mu = x^\mu + \xi^\mu, \\ \phi_i \to \phi_i'^\mu = \phi_i + \bar{\delta}\phi_i. \end{cases} \tag{C.2}$$

Here $\bar{\delta}\phi_i$ denotes the *intrinsic* change of the field. The notation $\delta\phi_i$ is reserved for *local* change, $\delta\phi_i(x) = \phi_i'(x) - \phi_i(x) = -\xi^\mu\partial_\mu\phi_i + \bar{\delta}\phi_i$. Take the action for the transformed fields and spacetime, expand to first order

$$\begin{aligned} S'[\phi_i'] &= \int_{M'} \mathrm{d}^4x'\, \mathcal{L}(\phi_i'^\mu, \partial_\mu\phi_i') = \int_{M'} \mathrm{d}^4x' \left[\mathcal{L}(\phi_i, \partial_\mu\phi_i) + \frac{\partial\mathcal{L}}{\partial_\mu\phi_i}\bar{\delta}\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\bar{\delta}\partial_\mu\phi_i\right] \\ &= \int_{M'} \mathcal{L}(\phi_i, \partial_\mu\phi_i)\,\mathrm{d}^4x' + \int_{M'} \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\bar{\delta}\phi_i\right)\mathrm{d}^4x' + \int_{M'} \left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)\bar{\delta}\phi_i\,\mathrm{d}^4x' \end{aligned} \tag{C.3}$$

To derive the second line, $\bar{\delta}\partial_\mu\phi_i = \partial_\mu\bar{\delta}\phi_i$ is used. For the first term

$$\int_{M'} \mathcal{L}(\phi_i, \partial_\mu\phi_i)\,\mathrm{d}^4x' = \int_M \mathcal{L}(\phi_i, \partial_\mu\phi_i)\,\mathrm{d}x^4 + \int_{\partial M} \mathrm{d}^3y\, n_\mu\xi^\mu\mathcal{L}(\phi_i, \partial_\mu\phi_i). \tag{C.4}$$

The first term is just $S[\phi]$, while the second is a compensation for replacing the integral over M' by that over M. For the second and third term in the last line of (C.3), such a replacing should induce a substitution of $\bar{\delta}\phi_i$ to $\delta\phi_i = -\xi^\mu\partial_\mu\phi_i + \bar{\delta}\phi_i$ since the fields should "feel" the change of spacetime. With this, invoking Stokes's theorem for the second term on the right hand side of (C.4) should give

$$S'[\phi_i'] = S[\phi_i] + \int_M \mathrm{d}^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \xi^\mu\mathcal{L}\right) + \int_M \mathrm{d}^4x\,\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)\delta\phi_i. \tag{C.5}$$

If the equation of motion is satisfied, the last term vanishes. If the infinitesimal transformation (C.2) is a symmetry, by definition $S' = S$, the second term vanishes as well. Define the integrand taking total divergence as $-\epsilon j^\mu$ where ϵ is the infinitesimal amount of transformation, we have the expression for the Noether current

$$\epsilon j^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i - \xi^\mu \mathcal{L}, \tag{C.6}$$

and the conclusion $\partial_\mu j^\mu = 0$ when (i) the transformation is a global symmetry and (ii) the field equations of motion are satisfied.

Notice that if the transformation (C.2) is solely due to a spacetime transformation (e.g., $\bar{\delta}\phi_i$ only comes from tensor transformation and vanish for scalar fields), then the local transformation $-\delta\phi$ is nothing but the component form of the Lie derivative of the fields along ξ^μ (see appendix C of [62]). Hence the Hodge dual of the metric dual of Noether current defined in (C.6) resembles its counterpart using forms

$$\boldsymbol{J}[\xi] = \boldsymbol{\Theta}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \boldsymbol{L}. \tag{C.7}$$

And if the on-shell condition for the fields is relaxed, the condition that (C.2) is a symmetry implies in (C.5), $-\partial_\mu j^\mu = E_{\phi_i} \mathcal{L}_\xi \phi_i$, again suggesting a resemblance of $\mathrm{d}\boldsymbol{J}[\xi] = -\boldsymbol{E}_\phi \mathcal{L}_\xi \phi$ derived in the main text. However, this resemblance should not be taken too seriously since the expresssion $\delta \boldsymbol{L} = \boldsymbol{E}_\phi \delta\phi + \mathrm{d}\boldsymbol{\Theta}(\phi, \delta\phi)$ cannot yet find a counterpart in this heuristic derivation in flat space classical field theory. Especially the corresponding expression for the symplectic potential $(n-1)$-form $\boldsymbol{\Theta}$ is still lacking (although the form of Noether current might suggest something).

C.2 Stokes's theorem

C.2.1 The first version

Theorem C.1. *Let M be an orientable n-dimensional manifold. Let V be a p-dimensional compact oriented submanifold with boundary ∂V. Let $\boldsymbol{\omega}$ be a $(p-1)$-form on M. Then*

$$\int_V \mathrm{d}\boldsymbol{\omega} = \int_{\partial V} \boldsymbol{\omega}. \tag{C.8}$$

Skeptch of the proof: Suppose $\boldsymbol{\beta} = i^* \boldsymbol{\omega}$ is the pull-back of $\boldsymbol{\omega}$ under the inclusion map $i : V \hookrightarrow M$. Introduce a partition of unity for the submanifold V. It suffices to prove that the identity for $\boldsymbol{\beta}$ holds in each patch of this partition of unity. If the patch under consideration does not overlap with ∂V, it only needs to be verified that both sides vanish. When the patch overlaps with ∂V, the non-zero contribution from the boundary on both sides agree, provided that the orientation when integrating forms is properly taken into account. Details of the proof are provided in [63].

The form of Stokes's theorem (C.8) is not very practical in terms of physical application. For definiteness from now on we take M to be an n-dimensional orientable manifold with boundary ∂M and $\boldsymbol{\omega}$ to be an $(n-1)$-form. Let g_{ab} be a metric of the manifold M and $\boldsymbol{\epsilon}$ be the volume (pseudo)n-

form defined from this metric. Suppose there exists a vector B^a, whose interior product with the volume form is $\boldsymbol{\omega}$, i.e., $i_{B^a}\boldsymbol{\epsilon} = B \cdot \boldsymbol{\epsilon} = \boldsymbol{\omega}$. The interior product defined abtractly in terms of a set of axioms is given in [63], where its existence and uniqueness are also shown. For all practical purposes, such a definition coincides with the usual "contraction with the first index" frequently used in physics text. In abstract index $(i_{B^a}\boldsymbol{\epsilon})_{b_1 b_2 \ldots b_{n-1}} = B^a \epsilon_{a b_1 \ldots b_{n-1}}$. Such a statement also amounts to $\star\boldsymbol{B} = \boldsymbol{\omega}$, where $\star$ denotes the Hodge dual and $\boldsymbol{B}$ is the metric dual of B^a. It implies that such a vector B^a always exists since it's just the Hodge dual of $\boldsymbol{\omega}$ up to a sign, because for any p-form $\boldsymbol{\alpha}$, $\star(\star\boldsymbol{\alpha}) = (-1)^{s+p(n-p)}\boldsymbol{\alpha}$, where s is the number of minuses in the metric signature. In coordinate,

$$\begin{aligned} i_{B^a}\boldsymbol{\epsilon} &= \boldsymbol{\epsilon}(B^a, \cdot, \ldots, \cdot) \\ &= \sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n (B^a, \cdot, \ldots, \cdot) \\ &= \sum_{\nu=1}^{n} \sqrt{|g|}\, (-1)^{\nu-1} B^\nu \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^\nu} \wedge \cdots \wedge \mathrm{d}x^n. \end{aligned} \tag{C.9}$$

An over hat means that the corresponding term is missing in the summand. The left hand side of (C.8) can be further computed as

$$\begin{aligned} \mathrm{d}(i_{B^a}\boldsymbol{\epsilon}) &= \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} (-1)^{\nu-1}\partial_\mu \left(\sqrt{|g|}\, B^\nu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^\nu} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{\nu=1}^{n} (-1)^{\nu-1}\partial_\nu \left(\sqrt{|g|}\, B^\nu\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^\nu} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \frac{1}{\sqrt{|g|}}\partial_\nu \left(\sqrt{|g|}\, B^\nu\right) \left(\sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n\right) \\ &= (\mathrm{div} B^a)\boldsymbol{\epsilon}, \end{aligned} \tag{C.10}$$

where Einstein summation rule is restored on the third line, and the last line comes from an alternative definition of the divergence of a vector $\mathrm{d}(i_{B^a}\boldsymbol{\epsilon}) \equiv (\mathrm{div} B^a)\boldsymbol{\epsilon}$ given in [63]. It's reassuring to see that it coincides with the well-known result

$$\nabla_\nu B^\nu = \frac{1}{\sqrt{|g|}}\partial_\nu \left(\sqrt{|g|}\, B^\nu\right) \tag{C.11}$$

frequently used in GR. Hence we have established

$$\text{l.h.s. of (C.8)} = \int_M (\nabla_a B^a)\boldsymbol{\epsilon} = \int_M \sqrt{|g|}\mathrm{d}^n x(\nabla_\nu B^\nu). \tag{C.12}$$

For the right hand side, suppose Gaussian normal coordinate is chosen. Let $\{x^1, x^2, \ldots, x^n\}$ be the coordinate of points in M in the vicinity of the boundary, and let $\{y^1, \ldots, y^{n-1}\} = \{x^2, \ldots, x^n\}$

be the coordinate of points on the boundary ∂M. The metric component in this coordinate system are $g_{11} = \pm 1$, $g_{1\mu} = 0$ for $\mu = 2, \dots n$, $g_{\mu\nu} = \gamma_{ij}$ for $\mu, \nu = 2, \dots n$ and $i = \mu - 1, j = \nu - 1$. γ_{ab} is the induced metric. In such a coordinate, $n_a = (\mathrm{d}x^1)_a$ is the normal 1-form to ∂M. Its metric dual is the normal vector to ∂M. To match the orientation of M and the one used in Stokes theorem, n^a is chosen to be "outward pointing" if it's spacelike and "inward pointing" if it's timelike[62, 63]. Then the last line of (C.9) gives

$$\begin{aligned}\int\limits_{\partial M} \boldsymbol{\omega} = \int\limits_{\partial M} i_{B^a}\boldsymbol{\epsilon} = \int\limits_{\partial M} \sqrt{|g|}\, B^1 \,\mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \\ = \int\limits_{\partial M} (\mathrm{d}x^1)_a B^a \sqrt{|\gamma|}\,\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{n-1} = \int\limits_{\partial M} \boldsymbol{n}(B)\hat{\boldsymbol{\epsilon}},\end{aligned} \tag{C.13}$$

where for the second equality only the first term in the sum contributes since there is no $\mathrm{d}x^1$ on ∂M. $\hat{\boldsymbol{\epsilon}}$ is the volume form on the boundary inherited from the induced metric γ_{ab} and by definition $\boldsymbol{\epsilon} = \boldsymbol{n} \wedge \hat{\boldsymbol{\epsilon}}$. The identity is proven in Gaussian normal coordinate. However, the starting point and result are both coordinate independent. If the identity holds in one frame, it should hold in any other coordinate frame. Hence we have established

$$\text{r.h.s. of (C.8)} = \int\limits_{\partial M} \boldsymbol{n}(B)\hat{\boldsymbol{\epsilon}} = \int\limits_{\partial M} \sqrt{|\gamma|}\mathrm{d}^{n-1}y\, n_\nu B^\nu. \tag{C.14}$$

To summarize, we now have the index-free representation, abstract index representation and component representation for the Stokes's theorem

$$\int\limits_{M} \mathrm{d}\boldsymbol{\omega} = \int\limits_{\partial M} \boldsymbol{\omega}, \tag{C.15a}$$

$$\int\limits_{M} (\nabla_a B^a)\epsilon_{b_1 \dots b_n} = \int\limits_{\partial M} (n_a B^a)\hat{\epsilon}_{b_1 \dots b_{n-1}}, \tag{C.15b}$$

$$\int\limits_{M} \sqrt{|g|}\,\mathrm{d}^n x (\nabla_\nu B^\nu) = \int\limits_{\partial M} \sqrt{|\gamma|}\,\mathrm{d}^{n-1}y\,(n_\nu B^\nu), \tag{C.15c}$$

where the 1-forms $\boldsymbol{B}$ and $\boldsymbol{n}$ are given by $\star\boldsymbol{B} = \boldsymbol{\omega}$ and $\boldsymbol{\epsilon} = \boldsymbol{n} \wedge \hat{\boldsymbol{\epsilon}}$. This dictionary bewteen the three representations can be further enriched through the study of conserved charges in GR or any diffeomorphism invariant gravity theory involving a metric and external matter fields.

C.2.2 The second version

There is a second version of Stokes's theorem, which is also frequently used in GR. Let Σ be a hypersurface with unit normal n^a in an n-dimensional manifold M. Suppose Σ has boundary $\partial\Sigma$ which is a codimension 2 submanifold of M. And the unit normal of $\partial\Sigma$ is N^a. To maintain generality while keeping the potential of application in GR, consider n^a to be either timelike or spacelike depending on whether M is Lorentzian or Riemannian, and N^a to be spacelike regardless. The orientation of the volume form and induced volume forms is chosen such that $\boldsymbol{\epsilon} = \boldsymbol{n} \wedge \hat{\boldsymbol{\epsilon}} = \boldsymbol{n} \wedge \boldsymbol{N} \wedge \tilde{\boldsymbol{\epsilon}}$, where $\boldsymbol{\epsilon}$, $\hat{\boldsymbol{\epsilon}}$ and $\tilde{\boldsymbol{\epsilon}}$ are volume forms for M, Σ and $\partial\Sigma$ respectively. Let $\boldsymbol{B}$ be a 2-form and B^{ab} be its metric dual. In component form, this second version of Stokes's theorem reads

$$\int\limits_{\Sigma} \nabla_\nu B^{\mu\nu} \mathrm{d}\Sigma_\mu = \frac{1}{2} \int\limits_{\partial\Sigma} B^{\mu\nu} \mathrm{d}S_{\mu\nu}, \tag{C.16}$$

where $\mathrm{d}\Sigma_\mu = n_\mu \sqrt{\gamma}\, \mathrm{d}^{n-1}y$ and $\mathrm{d}S_{\mu\nu} = 2n_{[\mu} N_{\nu]} \sqrt{q}\, \mathrm{d}^{n-2}z$, with γ_{ij} and q_{AB} being the component form of the induced metrics on Σ and $\partial\Sigma$. The proof of this formula presented here is adapted from [64]. The key is to find a convenient coordinate to establish the equality (C.16), as is done for establishing the dictionary (C.15) before. Since it's a tensorial equation, it must hold in any coordinate system.

In a neighborhood of Σ, choose Gaussian normal coordinate as before, such that $\{x^\mu\}$ with $\mu = 1, 2, \ldots, n$ labels the point in M, and $\{y^1, \ldots, y^{n-1}\} = \{x^2, \ldots, x^n\}$ are intrinsic coordinates on Σ. In this coordinate $n^\mu = (\pm 1, 0, \ldots, 0)$, and $\sqrt{|g|} = \sqrt{\gamma}$. Since an integration over the entire Σ is involved, choosing Gaussian normal coordinate in a neighborhood of $\partial\Sigma$ is certainly not enough. Rather, "a stack of Gaussian normal coordinates" should be chosen. View Σ now as a "volume" in its own right instead of as a "surface" in M, choose y^1 to be a radial coordinate in Σ such that $y^1 = \text{const}$ determines a set of submanifolds Φ_{y^1} satisfying

$$\bigcup_{y^1=0}^{y^1=1} \Phi_{y^1} = \Sigma. \tag{C.17}$$

Moreover, $\Phi_{y^1=0}$ is the "zero volume point" in Σ and $\Phi_{y^1=1}$ is $\partial\Sigma$. Let $\{z^1, \ldots, z^{n-2}\} = \{x^3, \ldots, x^n\}$ be the coordinates on $\partial\Sigma$, with the fixed value of $x^2 = y^1 = 1$. Work on coordinate $\{x^\mu\}$, $\{x^i\}$, $\{x^C\}$ respectively in M, Σ, $\partial\Sigma$ with $\mu = 1, \ldots, n$, $i = 2, \ldots, n$ and $C = 3, \ldots, n$. In this coordinate $N^\mu = (0, 1, \ldots, 0)$ and $\sqrt{|g|} = \sqrt{\gamma} = \sqrt{q}$.

Now, the left hand side of (C.16) is actually the integration of $\star\boldsymbol{A}$, with $\boldsymbol{A}$ being the 1-form with component $A_\mu = \nabla^\nu B_{\mu\nu}$. (C.8) suggests that if there exists an $(n-2)$-form $\boldsymbol{\beta}$ such that

$$\int\limits_{\Sigma} \star\boldsymbol{A} = \int\limits_{\Sigma} \mathrm{d}\boldsymbol{\beta}, \tag{C.18}$$

then the integral can be written as an integral of $\boldsymbol{\beta}$ on the boundary $\partial\Sigma$. Write down an ansatz

$$\boldsymbol{\beta} = b(z)\sqrt{q}\mathrm{d}x^3 \wedge \cdots \wedge \mathrm{d}x^n. \tag{C.19}$$

(In fact the only guess can be $\star\boldsymbol{B}$, which turns out to be indeed the case) Computing the integral on the left hand side of (C.16) in the aforementioned coordinate system will give the scalar function b. Using the component expression of n_μ (ignoring the ± 1 which is to be fixed in specifc contexts), a generalization of the divergence formula (C.11)

$$\nabla_\nu B^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}B^{\mu\nu}\right), \tag{C.20}$$

as well as $\sqrt{|g|} = \sqrt{\gamma} = \sqrt{q}$, we can get

$$\begin{aligned}\int_\Sigma \nabla_\nu B^{\mu\nu}\mathrm{d}\Sigma_\mu &= \int_\Sigma \partial_i\left(\sqrt{\gamma}B^{1i}\right)\mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \\ &= \int_\Sigma \partial_2\left(\sqrt{\gamma}B^{12}\right)\mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n + \int_\Sigma \partial_C\left(\sqrt{\gamma}B^{1C}\right)\mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \\ &= \int \sqrt{\gamma}B^{12}\mathrm{d}x^3 \wedge \cdots \wedge \mathrm{d}x^n\Big|_{x^2=0}^{x^2=1} + \sum_C \int \sqrt{\gamma}B^{12}\mathrm{d}x^2 \wedge \ldots \widehat{\mathrm{d}x^C} \wedge \mathrm{d}x^n\Big|_{x^C_{\text{ini}}}^{x^C_{\text{fin}}} \\ &= \int_{\partial\Sigma=\Phi_1} B^{12}\sqrt{q}\,\mathrm{d}x^3 \wedge \cdots \wedge \mathrm{d}x^n.\end{aligned} \tag{C.21}$$

On the third line, the second term vanishes because x^C are angular coordinates in this local chart. For example in spherical coordinate in flat space, $x^C_{\text{ini}} = x^C_{\text{fin}}$ for azimuthal angles. And for polar angles, e.g., θ, the $\sin\theta$ in the determinant $\sqrt{\gamma}$ gives 0 when $\theta = 0$ or $\theta = \pi$. It is assumed that the generalization to a generic $(n-1)$-dimensional manifold should hold provided that $x^2 = y^1$ in our coordinate choice is well justified as a radial coordinate in the sense of (C.17). The last line made use of the fact that $x^2 = 0$ is a "point" with zero $(n-2)$-volume. Now match the result with the integration of the ansatz (C.19), noticing that $(\partial_1)^a = n^a$ and $(\partial_2)^a = N^a$ in this coordinate basis,

$$\int_{\partial\Sigma} \boldsymbol{B}(n^a, N^a)\sqrt{q}\,\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^{n-2} = \int_{\partial\Sigma}\boldsymbol{\beta}, \tag{C.22}$$

$$\boldsymbol{\beta} = \boldsymbol{B}(n^a, N^a)\sqrt{q}\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^{n-2} = B^{\mu\nu}n_\mu N_\nu\tilde{\boldsymbol{\epsilon}} = \frac{1}{2}B^{\mu\nu}(n_\mu N_\nu - n_\nu N_\mu)\tilde{\boldsymbol{\epsilon}} = \frac{1}{2}B^{\mu\nu}\mathrm{d}S_{\mu\nu} \tag{C.23}$$

In the third equality of (C.23) the anti-symmetry of $B^{\mu\nu}$ is used. Hence we have reached our goal:

proving (C.16) in a convenient coordinate system, and arguing it is true since it's a tensorial equation.

However, one might worry about the availability of such a coordinate system, especially the existence of a radial coordinate y^1 on Σ to make the Σ-hypersurfaces $y^1 = C$ possess the topology of an $(n-2)$-sphere $\mathbb{S}^{n-2}$. A more rigorous proof can be pursued in analogy of the proof of Stokes's theorem (C.8) in [63]. In fact, to rigorously define the integration of a p-form, an oriented parametrized p-subset in $\mathbb{R}^p$ has to be introduced, as well as a partition of unity subodinate to a finite cover of the integration region on the manifold. The former is not mentioned before since it is assumed that the coordinate map already provides a natural oriented parametrized p-subset. As for the latter, it turns out that one has to compute the forms weighted by the partition of unity. In the case of proving (C.16), we can introduce a finite cover of Σ: $\bigcup_\alpha V_\alpha = \Sigma$, with a partition of unity $\{f_\alpha\}$, which is a set of functions $f_\alpha : \Sigma \to \mathbb{R}$ satisfying: (i) $f_\alpha \geq 0$, (ii) $\mathrm{supp}(f_\alpha) = V_\alpha$, (iii) $\forall p \in \Sigma,\ \sum_\alpha f_\alpha(p) = 1$. The integral on the left hand side of (C.16) should be defined as

$$\int\limits_\Sigma \star \boldsymbol{A} := \sum_\alpha \int\limits_{V_\alpha} f_\alpha \star \boldsymbol{A}. \tag{C.24}$$

When reaching a step similar to (C.21), only the V_α's satisfying $V_\alpha \bigcap \partial\Sigma \neq \emptyset$ will contribute. Notice that the finite cover $\{V_\alpha\}$ does not at all have to be arranged in an "onion-shape" as the coordinate construction above. This justifies, in another way, why only the boundary term remains in the calculation of (C.21). In some sense, this formal argument involving a proper mathematical definition of form integration allows us to avoid the difficulty of constructing "a stack of Gaussian normal coordinates", so that we can again just construct such a coordinate in a neighborhood of $\partial\Sigma$ to proceed, in much the same spirit of establishing the dictionary (C.15).

To summarize, we again have a dictionary between abstract and component expressions for this second version of Stokes's theorem for a given 2-form $\boldsymbol{B}$:

$$\int\limits_\Sigma \star \boldsymbol{A} = \int\limits_{\partial\Sigma} \star \boldsymbol{B}, \tag{C.25a}$$

$$\int\limits_\Sigma \nabla_b B^{ab} \epsilon_{ac_1\ldots c_{n-1}} = \frac{1}{2} \int\limits_{\partial\Sigma} B^{ab} \epsilon_{abc_1\ldots c_{n-2}}, \tag{C.25b}$$

$$\int\limits_\Sigma \nabla_\nu B^{\mu\nu} n_\nu \sqrt{\gamma}\, \mathrm{d}^{n-1}y = \int\limits_{\partial\Sigma} B^{\mu\nu} n_\mu N_\nu \sqrt{q}\, \mathrm{d}^{n-2}z = \frac{1}{2} \int\limits_{\partial\Sigma} B^{\mu\nu}\, \mathrm{d}S_{\mu\nu}. \tag{C.25c}$$

In (C.25a), $\boldsymbol{A}$ is a 1-form derived from $\boldsymbol{B}$ by the abstract notation $A_a \equiv \nabla^b B_{ab}$. The orientation for the volume forms and unit normals are chosen such that $\boldsymbol{\epsilon} = \boldsymbol{n} \wedge \hat{\boldsymbol{\epsilon}} = \boldsymbol{n} \wedge \boldsymbol{N} \wedge \tilde{\boldsymbol{\epsilon}}$, where $\boldsymbol{\epsilon}$, $\hat{\boldsymbol{\epsilon}}$ and $\tilde{\boldsymbol{\epsilon}}$ are volume forms for M, Σ and $\partial\Sigma$ respectively. n^a is again outward-pointing if spacelike and inward-pointing if timelike. This is the source of sign difference of (C.25b) with many GR text since it's conventional to pick n^a to be future-pointing and it's the $-n^a$ that should enter in all the formulae above.

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