# A triad formulation of rigid body mechanics

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## 1 Summary of results

Since later I want to discuss the bound (and most interestingly circular) orbit of two spinning rigid spheres in Newtonian gravity, in this note some generalities of rigid body motion are established. Contrary to the traditional text which mainly use Euler angles to describe the rotational (spinning) degrees of freedom such as [1, 2], this note introduces a Euclidean orthonormal triad, such that comparisons and generalization to relativistic mechanics can be stated. Moreover, the distinction between the inertial background frame and body-fixed frame is made clear by the novel mathematical description. This was somewhat confusing or elusive to me when I first learned it as an undergraduate.

Notation: middle Lattin indice $i, j, k \dots$ always refer to the components in the background frame, and early Lattin indices with hat $\hat{a}, \hat{b}, \hat{c} \dots$ always refer to components in the rotating body frame. The early Lattin letters with no hat $a, b, c \dots = \{1, 2, 3\}$ refer to the labels of the background frame basis vectors. Boldface letters refer to abstract tensors, whose type should be specified in the context, such as basis vectors $\boldsymbol{e}_a$, $a = \{1, 2, 3\}$ in the background frame which has components $e_a^i = \delta_a^i$. If the boldface letter is a form, a superscript is sometimes used to specify its rank, such as $\boldsymbol{\omega}^{(2)}$. The comparison of classical and relativistic mechanics is summarised in the following table

| | classical mechanics | relativistic mechanics |
|---|---|---|
| triad/tetrad | $e_{\hat{a}}^i(t)$ | $e_A^\mu(\tau)$ |
| orthonormality 1 | $\delta_{ij} e_{\hat{a}}^i e_{\hat{b}}^j = \delta_{\hat{a}\hat{b}}$ | $g_{\mu\nu} e_A^\mu e_B^\nu = \eta_{AB}$ |
| orthonormality 2 | $\delta_{\hat{a}\hat{b}} e_i^{\hat{a}} e_j^{\hat{b}} = \delta_{ij}$ | $\eta_{AB} e_\mu^A e_\nu^B = g_{\mu\nu}$ |
| evolution | $e_{\hat{a}}^i = R^i{}_j(t) e_{\hat{a}}^j(0)$, $R^i{}_j(t) \in \mathrm{SO}(3)$ | $e_A^\mu(\tau) = \Lambda^\mu{}_\nu(\tau)\epsilon_A^\nu$, $\Lambda^\mu{}_\nu(\tau) \in \mathrm{SO}(1,3)$ |
| angular velocity 2-form | $e_i^{\hat{a}} \dot{e}_{\hat{a}j} = \omega_{ij}$ | $e_\mu^A \dot{e}_{A\nu} = \Omega_{\mu\nu}$ |
| linear momentum | $P_i = \frac{\partial L}{\partial \dot{R}^i}$ | $p_\mu = \frac{\partial L}{\partial u^\mu}$ |
| (spin) angular momentum 2-form | $S_{ij} = 2\frac{\partial L}{\partial \omega^{ij}}$ | $S_{\mu\nu} = 2\frac{\partial L}{\partial \Omega^{\mu\nu}}$ |
| equations of motion | $\dot{P}_i = \frac{\partial L}{\partial R^i} \equiv F_i$, $\dot{S}^i = \frac{\partial L}{\partial \varphi_i} \equiv N^i$ | $\dot{p}_\mu = \frac{1}{2} R_{\rho\sigma\nu\mu} S^{\rho\sigma} u^\nu$, $\dot{S}_{\mu\nu} = 2p_{[\mu} u_{\nu]}$ |

Table 1: The rightmost column contains the well-known results taken from either [3] (for special relativistic case) or [4] (for general relativistic case). Instead of taking these results for granted, in this note the formulae in the column of classical mechanics are either motivated from definition, or derived from first principle. It should be pointed out that orthonormality 1 is given from definition, but orthonomality 2, for the classical case, is derived from property of SO(3).

With this classical correspondence in mind, it's less likely to get lost in the mathematics as we proceed in the relativistic calculations. I was indeed a bit lost when trying to use the spinning particle dynamics in curved spacetime to reduce the tensorial first law to scalar form. This note (and the few next to come) is to provide a clearer physical picture by going back to the classical world which is easier to imagine and visualize.

# 2 Kinematics

## 2.1 triad and angular velocity

The first task is to come up with mathematical description of the configuration of the system. There are 6 degrees of freedom for a rigid body: 3 translational and 3 rotational. We can first build a fixed background coordinate, and describe the configuration of the rigid body in terms of the coordinate of a reference point, and the relative position of the points on the rigid body with respect to the reference point. The former encodes the translational degrees of freedom, and the latter encodes the rotational ones. The basis vectors of the background frame $\boldsymbol{e}_a,\ a = \{1, 2, 3\}$ are time independent, while those for the body frame $\boldsymbol{e}_{\hat{a}},\ \hat{a} = \{\hat{1}, \hat{2}, \hat{3}\}$ are. Their relation is given by

$$\boldsymbol{e}_{\hat{a}} = \mathscr{R}(t)\boldsymbol{e}_a, \text{ or in components } e^i_{\hat{a}} = R^i{}_j(t)e^i_a. \tag{1}$$

Here $\mathscr{R} \in \mathrm{SO}(3)$ with matrix components $R^i{}_j$. The usual convention $R^i{}_j(0) = \delta^i{}_j$ means that at $t = 0$, the basis vectors for the two frames coincide. By definition $\delta_{ij}e^i_{\hat{a}}(t)e^j_{\hat{b}}(t) = \delta_{\hat{a}\hat{b}}$, meaning the triad stays orthonormal at all times, although they are in general constantly rotating. A computation of $\delta^{\hat{a}\hat{b}}e^i_{\hat{a}}e^j_{\hat{b}}$ shows

$$\begin{aligned}
\delta^{\hat{a}\hat{b}}e^i_{\hat{a}}e^j_{\hat{b}} &= e^i_{\hat{1}}e^j_{\hat{1}} + e^i_{\hat{2}}e^j_{\hat{2}} + e^i_{\hat{3}}e^j_{\hat{3}} \\
&= (R^i{}_{i_1}e^{i_1}_1)(R^j{}_{j_1}e^{j_1}_1) + (R^i{}_{i_2}e^{i_2}_2)(R^j{}_{j_2}e^{j_2}_2) + (R^i{}_{i_3}e^{i_3}_3)(R^j{}_{j_3}e^{j_3}_3) \\
&= (R^i{}_{i_1}\delta^{i_1}_1)(R^j{}_{j_1}\delta^{j_1}_1) + (R^i{}_{i_2}\delta^{i_2}_2)(R^j{}_{j_2}\delta^{j_2}_2) + (R^i{}_{i_3}\delta^{i_3}_3)(R^j{}_{j_3}\delta^{j_3}_3) \\
&= R^i{}_1R^j{}_1 + R^i{}_2R^j{}_2 + R^i{}_3R^j{}_3 \\
&= (\mathscr{R}\mathscr{R}^{\mathrm{T}})^{ij} = \delta^{ij}.
\end{aligned} \tag{2}$$

We now have the first three rows for Table 1. As is mentioned in the caption, the orthonormality 2 is derived, not given.

Call the body reference point $O$, which does not necessarily have to be the center of mass (COM) for the moment. A given point $A$ on the rigid body is described by the position vector

$$\boldsymbol{R}_A = \boldsymbol{R} + \boldsymbol{r}_A = \boldsymbol{R} + A^{\hat{a}}\boldsymbol{e}_{\hat{a}}, \text{ or in components, } R^i_A = R^i + A^{\hat{a}}e^i_{\hat{a}} \tag{3}$$

The configuration space is thus

$$\mathcal{C} = \{\boldsymbol{R}(t), \mathscr{R}(t)\}, \text{ or } \mathcal{C} = \{\boldsymbol{R}(t), \boldsymbol{e}_{\hat{a}}(t)\}. \tag{4}$$

This means that at any instant $t$, the body is characterized by the position of the reference point, and a rotation from the background axes to the body axes about that point. A fixed point $A$ on the body is enough to determine all the other points since the relative position between points of a rigid body is fixed a priori. Hence the components $A^{\hat{a}}$ is time independent. Figure 1 shows this physical setup at an initial moment $t = 0$ and an arbitrary instant. The velocity of point $A$ at time $t$ is

$$\dot{\boldsymbol{R}}_A = \dot{\boldsymbol{R}} + A^{\hat{a}}\dot{e}_{\hat{a}}. \tag{5}$$

The first term, the translational velocity of the reference point, is trivial. The second term motivates the definition of the angular velocity. To see this, consider an infinitesimal time internval from $t = 0$,

$$r_{Aj}(dt) = A^{\hat{a}}R_{jk}(0 + dt)e^k_a \equiv A^{\hat{a}}\left(\delta_{jk} + dt\,\omega^i(L_i)_{jk}\right)e^k_a. \tag{6}$$

The second equality comes from the theory for SO(3) in fundamental representation[1]. It can be regarded as a definition of the angular velocity, i.e., the vector which multiplied by $dt$ gives the infinitesimal amount of generation of rotation[2]. Adopting an active viewpoint of transformation, in

[1]Notice here we write $e^i_a$ instead of $e^i_{\hat{a}}(0)$ just for convenience. Exceptionally $a$ and $\hat{a}$ should be summed over here.

[2]Strictly speaking it is a pseudo-vector because of its behavior under spatial parity transformation. But we don't distinguish the difference here, nor for its dual pseudo-2-form, which is considered to be more fundamental by some mathematical physicists.

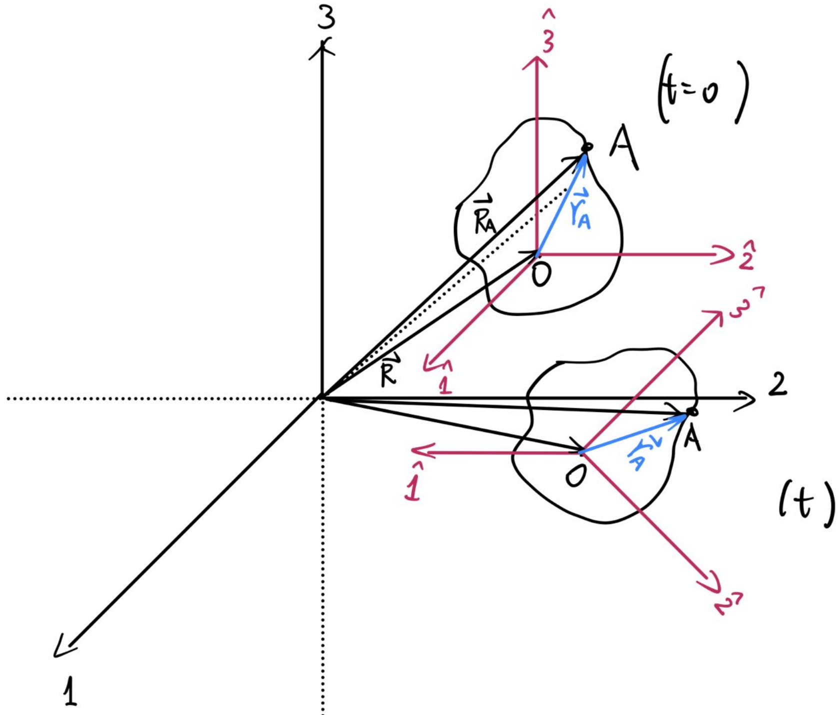


Figure 1: The illustration of coordinate choice and rigid body motion. At $t = 0$, the basis vectors of the background frame and body frame coincide.

fundamental representation the SO(3) generators $(L_i)_{jk}$, meaning the $jk$ matrix component of the $i$th generator, takes the form

$$(L_i)_{jk} = -\epsilon_{ijk}, \text{ satisfying } [L_i, L_j] = \epsilon_{ijk} L_k. \tag{7}$$

Plug it back into (6), we have

$$r_{Aj}(dt) = r_{Aj}(0) + dt\epsilon_{jik}\omega^i e^k_a A^{\hat{a}} = r_{Aj}(0) + dt(\boldsymbol{\omega} \times \boldsymbol{r}_A(0))_j. \tag{8}$$

This yields, in component free form,

$$\dot{\boldsymbol{r}}_A(0) = \boldsymbol{\omega} \times \boldsymbol{r}_A(0). \tag{9}$$

Of course this argument can be generalized to any instant of time, provided that the Taylor expansion of $R_{jk}(t)$ at any time gives

$$R_{jk}(t + dt) = R_{jk}(t) + dt\, \boldsymbol{\omega} \cdot \boldsymbol{L}_{jl} R^l{}_k(t). \tag{10}$$

Apply this argument to the body frame basis vector itself

$$\dot{e}_{\hat{a}j} = \omega^i (L_i)_{jk} e^k_{\hat{a}} = \omega^i \epsilon_{ikj} e^k_{\hat{a}} \equiv \omega_{kj} e^k_{\hat{a}}, \tag{11}$$

where $\omega_{kj}$ is the dual 2-form of the angular velocity 1-form. The orthonormality condition yields

$$e^{\hat{a}}_k \dot{e}_{\hat{a}j} = \omega_{kj}. \tag{12}$$

We have therefore established the rows for "evolution" and "angular velocity 2-form" in Table 1.

At any time, the velocity of the point $A$ on the rigid body is given by

$$\dot{\boldsymbol{R}}_A = \dot{\boldsymbol{R}} + \dot{\boldsymbol{r}}_A \equiv \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{r}_A. \tag{13}$$

The intepretation is clear: at any time the velocity is the superposition of a translational velocity of the reference point and a rotational velocity relative to that point. The angular velocity is the rate of change of rotation generation by the generators in $\mathfrak{so}(3)$.

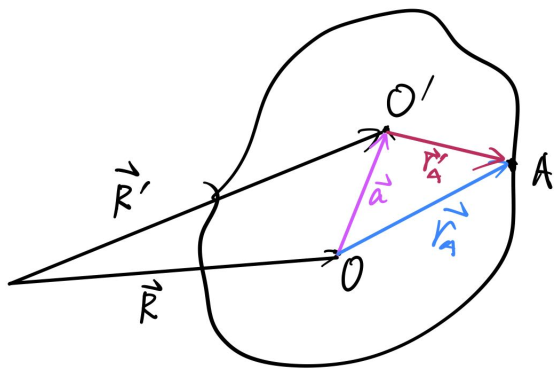


Figure 2: Illutration of the change of reference point.

Notice that another point $O'$ can be picked as the reference point, with $\boldsymbol{a}$ being the vector pointing from $O$ to $O'$, and $\boldsymbol{r}_A = \boldsymbol{a} + \boldsymbol{r}'_A$. See Fig 2 for an illustration. The new translational velocity and angular velocity are

$$\boldsymbol{V}' = \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{a}, \quad \boldsymbol{\omega}' = \boldsymbol{\omega}. \tag{14}$$

(See (31.3) of Landau&Lifshitz[1], and derivations therein) It means that the angular velocity does not depend on the reference point, and that the translational velocity of the new reference point is given by a superposition of translational velocity and rotational velocity of the new point relative to the old one.

We have established for a given point $A$ on the rigid body with position vector $\boldsymbol{r}_A$ relative to the reference point, that

$$\frac{d}{dt}\boldsymbol{r}_A = A^{\hat{a}} \frac{d}{dt}\boldsymbol{e}_{\hat{a}} = A^{\hat{a}}(\boldsymbol{\omega} \times \boldsymbol{e}_{\hat{a}}) = \boldsymbol{\omega} \times \boldsymbol{r}_A. \tag{15}$$

But for a generic vector $\boldsymbol{B}$, not necessarily the position vector of a point in the body (for example the position of an ant crawling on the body), we generally have $\boldsymbol{B} = B^{\hat{a}}(t)\boldsymbol{e}_{\hat{a}}(t)$. The component in the body frame also depends on time. In this case

$$\frac{d}{dt}\boldsymbol{B} = \dot{B}^{\hat{a}}\boldsymbol{e}_{\hat{a}} + B^{\hat{a}}(\boldsymbol{\omega} \times \boldsymbol{e}_{\hat{a}}) \equiv \frac{\hat{d}}{dt}\boldsymbol{B} + \boldsymbol{\omega} \times \boldsymbol{B}, \tag{16}$$

where $\hat{d}/dt$ is defined as the time derivative in the body frame. This is (4.86) in [2], and (36.1) in [1]. The distinction between the two time derivatives in background and body frame is now much clearer than explained by words. The Euler equation of motion for the rigid body is always the equation of motion in the body frame (often in terms of the Euler angles). Hence the Lattin indices in traditional text like [1, 2] should correspond to hatted early Lattin indices in this note.

## 2.2 Kinetic energy and inertia tensor

If we want the dynamics, we want to be able to write down the Lagrangian $L = T - V$. Formally the Lagrangian is a function with dependence on the generalized coordinates and their velocities $L(\boldsymbol{R}, \dot{\boldsymbol{R}}, \boldsymbol{e}_{\hat{a}}, \dot{\boldsymbol{e}}_{\hat{a}})$. We know that $\dot{\boldsymbol{e}}_{\hat{a}} = \boldsymbol{\omega} \times \boldsymbol{e}_{\hat{a}}$. Equivalently we can write $L(\boldsymbol{R}, \dot{\boldsymbol{R}}, \boldsymbol{\varphi}, \boldsymbol{\omega})$, with $\boldsymbol{\varphi}$ being the rotation angles that $\boldsymbol{\omega}$ is conjugate to. The problem is, there is no such thing as the rotation angles being a state variable that enters the Lagrangian function. The rotation angles can only acquire an infinitesimal meaning as analysed above. If one insists on choosing some angles to incorporate the rotational configuration, one has to introduce a certain convention due to the non-abelian nature of SO(3). Euler angles is one such convention but we'll not put much emphasis on them in this note.

The first task is to write down the kinetic energy of the rigid body. For convenience, we can think of the rigid body as a set of discrete points. The generalization to continuum case is straightforward. The kinetic energy of the body is

$$T = \sum_A \frac{1}{2} m_A \dot{\boldsymbol{R}}_A^2 = \sum_A \frac{1}{2} m_A \left[\boldsymbol{V}^2 + (\boldsymbol{\omega} \times \boldsymbol{r}_A)^2 + 2\boldsymbol{V} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_A)\right]. \tag{17}$$

From now on we choose $O$ to be COM, which makes the last term vanish. The expression for the kinetic energy becomes

$$T = T_{trans} + T_{rot} = \frac{1}{2}m\boldsymbol{V}^2 + \sum_A \frac{1}{2}m_A(\boldsymbol{\omega} \times \boldsymbol{r}_A)^2, \tag{18}$$

where $m = \sum_A m_A$ is the total mass of the body. For the rotational kinetic energy, write it out in component form, use the identity

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{19}$$

for the Levi-Civita symbol, it gives

$$T_{rot} = \sum_A \frac{1}{2}m_A \left[\omega^2 r_A^2 - (\boldsymbol{\omega} \cdot \boldsymbol{r}_A)^2\right]. \tag{20}$$

Again write this in component form, after some algebra,

$$T_{rot} = \frac{1}{2}\omega^i\omega^j \left[\sum_A m_A \left(r_A^2\delta_{ij} - r_{Ai}r_{Aj}\right)\right] \equiv \frac{1}{2}\omega^i I_{ij}\omega^j = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}. \tag{21}$$

$\boldsymbol{I}$ is thus defined as the moment of inertia tensor (henceforth called inertia tensor for brevity). It's a symmetric (0,2) type tensor. Its components in the background frame is defined above as

$$I_{ij} = \sum_A m_A \left(r_A^2\delta_{ij} - r_{Ai}r_{Aj}\right), \tag{22}$$

or for continuum system

$$I_{ij} = \int \rho \mathrm{d}V(r^2\delta_{ij} - r_i r_j). \tag{23}$$

$I_{ij}$ is time dependent in the background frame because $\boldsymbol{r}_A$ for each point is. However, its component in the body frame is time independent. And the geometric meaning and symmetry arguments concerning principle axes only apply in the body frame. Hence it's much more desirable to talk about the inertia tensor in the body frame. In fact, this is what Landau and Lifshitz do[1], especially in their section 35 about Euler angles. Their Lattin indices in section 35 should correspond to the hatted indices in this note.

Let's first examine how the component of a vector transforms. Consider a generic vector $\boldsymbol{B}$, when expanded in the body frame basis $\boldsymbol{B} = B^{\hat{a}}\boldsymbol{e}_{\hat{a}}$. Simply take its $i$th component in the background frame, it gives $B^i = B^{\hat{a}}e^i_{\hat{a}}$. Using the orthonormality condition 1, contract both side with $e_{\hat{b}i}$, we can get $B_{\hat{b}} = B^i e_{\hat{b}i}$. So in addition to capturing the rotational configuration, the triad also allows us to switch between background frame and body frame for the vector components

$$B^i = B^{\hat{a}}e^i_{\hat{a}}, \quad B_{\hat{b}} = B^i e_{\hat{b}i}. \tag{24}$$

Then for the rank 2 covariant inertia tensor $\boldsymbol{I}$,

$$I_{\hat{a}\hat{b}} = I_{ij}e^i_{\hat{a}}e^j_{\hat{b}} = \int \rho \mathrm{d}V \left[r^2 e^i_{\hat{a}}e^j_{\hat{b}}\delta_{ij} - (e^i_{\hat{a}}r_i)(e^j_{\hat{b}}r_j)\right] = \int \rho \mathrm{d}V(r^2\delta_{\hat{a}\hat{b}} - r_{\hat{a}}r_{\hat{b}}). \tag{25}$$

The expression of the inertia tensor components stays the same in body frame. Since the body frame basis coincide with the background basis at $t = 0$, $I_{\hat{a}\hat{b}}$ computed at $t = 0$ will keep the same values at all times because the components $r_{\hat{a}}$ of position vector remains the same in the body frame. This is what Landau and Lifshitz used to compute the inertia tensor in section 32 of [1]. Choosing the body frame axes to be the principle axes will diagonalize $I_{\hat{a}\hat{b}}$, thereby simplifying the calculations.

We'll work in the background frame since we want to track both translational and spinning motion for our binary system later. Typically the potential of the system does not depend on the velocities. If we assume this, the Lagrangian is

$$L(\boldsymbol{R}, \dot{\boldsymbol{R}}, \boldsymbol{\varphi}, \boldsymbol{\omega}) = \frac{1}{2}m\boldsymbol{V}^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega} - V(\boldsymbol{R}, \boldsymbol{\varphi}). \tag{26}$$

The dependence on the angle variable $\boldsymbol{\varphi}$ is kept formally even if it only acquires an infinitesimal meaning. However, if in some specific scenario the potential depends explicitly on the triad, one might write the Lagrangian with functional dependence $L(\boldsymbol{R}, \dot{\boldsymbol{R}}, \boldsymbol{e}_{\hat{a}}, \dot{\boldsymbol{e}}_{\hat{a}})$, by rewriting the rotational kinetic energy in terms of $\boldsymbol{e}_{\hat{a}}$.

$$\begin{aligned} T_{rot} &= \sum_A \frac{1}{2} m_A \dot{\boldsymbol{r}}_A^2 = \sum_A \frac{1}{2} m_A (A^{\hat{a}} \dot{e}_{\hat{a}})^2 = \sum_A \frac{1}{2} m_A A^{\hat{a}} A^{\hat{b}} \dot{e}^i_{\hat{a}} \dot{e}^j_{\hat{b}} \delta_{ij} \\ &= \frac{1}{2} \dot{e}^i_{\hat{a}} \dot{e}^j_{\hat{b}} \left( \sum_A m_A A^{\hat{a}} A^{\hat{b}} \delta_{ij} \right) \equiv \frac{1}{2} \dot{e}^i_{\hat{a}} \dot{e}^j_{\hat{b}} \mathcal{I}^{\hat{a}\hat{b}}_{ij}. \end{aligned} \tag{27}$$

$\mathcal{I}^{\hat{a}\hat{b}}_{ij}$ is a new concept proposed in this note. It can be dubbed "the second form of inertia tensor". Its expression, in both discrete and continuum system is

$$\mathcal{I}^{\hat{a}\hat{b}}_{ij} = \sum_A m_A A^{\hat{a}} A^{\hat{b}} \delta_{ij}, \tag{28a}$$

$$\mathcal{I}^{\hat{a}\hat{b}}_{ij} = \int \rho \, r^{\hat{a}} r^{\hat{b}} \delta_{ij} \, \mathrm{d}V. \tag{28b}$$

It is useful to regard it as a (0,2) tensor in the background frame while treating $\hat{a}, \hat{b}$ as labels. The Lagrangian now becomes

$$L(\boldsymbol{R}, \dot{\boldsymbol{R}}, \boldsymbol{e}_{\hat{a}}, \dot{e}_{\hat{a}}) = \frac{1}{2} m \boldsymbol{V}^2 + \frac{1}{2} \dot{\boldsymbol{e}}_{\hat{a}} \cdot \boldsymbol{\mathcal{I}}^{\hat{a}\hat{b}} \cdot \boldsymbol{e}_{\hat{b}} - V(\boldsymbol{R}, \boldsymbol{e}_{\hat{a}}). \tag{29}$$

Checking the expression for the rotational kinetic energy (27) agrees with the first formula (21) is an educative exercise to get familiar with triad formalism.

$$\begin{aligned} T_{rot} &= \frac{1}{2} \int \rho dV r^{\hat{a}} r^{\hat{b}} \dot{e}^i_{\hat{a}} \dot{e}^j_{\hat{b}} \delta_{ij} = \frac{1}{2} \int \rho dV r^{\hat{a}} r^{\hat{b}} \dot{e}^i_{\hat{a}} \dot{e}^j_{\hat{b}} \, e^{\hat{c}}_i e_{\hat{c}j} \\ &= \frac{1}{2} \int \rho dV r^{\hat{a}} r^{\hat{b}} \left( e^{\hat{c}}_i \dot{e}^i_{\hat{a}} \right) \left( e_{\hat{c}j} \dot{e}^j_{\hat{b}} \right) \\ &= \frac{1}{2} \int \rho dV r^{\hat{a}} r^{\hat{b}} \left[ \left( e^{\hat{c}}_i e^i_{\hat{a}} \right)^{\cdot} - \dot{e}^{\hat{c}}_i e^i_{\hat{a}} \right] \left[ \left( e_{\hat{c}j} e^j_{\hat{b}} \right)^{\cdot} - \dot{e}_{\hat{c}j} e^j_{\hat{b}} \right] \\ &= \frac{1}{2} \int \rho dV \left( r^{\hat{a}} e^i_{\hat{a}} \right) \left( r^{\hat{b}} e^j_{\hat{b}} \right) \dot{e}^{\hat{c}}_i \dot{e}_{\hat{c}_j} = \frac{1}{2} \int \rho dV r^i r^j \dot{e}^{\hat{a}}_i \dot{e}_{\hat{b}} + \underbrace{\delta_{\hat{a}\hat{b}}}_{e^k_{\hat{a}} e_{\hat{b}k}} \\ &= \frac{1}{2} \int \rho dV r^i r^j \left( \delta^{kl} e_{\hat{a}l} \dot{e}^{\hat{a}}_i \right) \left( e_{\hat{b}k} \dot{e}^{\hat{b}}_j \right) = \frac{1}{2} \int \rho dV r^i r^j \delta^{kl} \omega_{li} \omega_{kj} \\ &= \frac{1}{2} \int \rho dV r^i r^j \delta^{kl} \left( \omega^m \varepsilon_{mli} \right) \left( \omega^n \varepsilon_{nkj} \right) \\ &= \frac{1}{2} \int \rho dV r^i r^j \omega^m \omega^n \varepsilon_{kmi} \varepsilon_{knj} \\ &= \frac{1}{2} \int \rho dV (\boldsymbol{\omega} \times \boldsymbol{r})^2 \end{aligned} \tag{30}$$

This is exactly the second term in (18), also the expression in (20), which leads to the rotational kinetic energy (21) expressed in the inertia tensor of "the first form". The derivation made use of the orthonormality conditions of the triad and the definition of the angular velocity 2-form.

If we have chosen another point as the reference point instead of the COM, what will be the change of inertia tensor and the form of total kinetic energy? As in section 2.1, if we choose point $O'$ to be the new reference point, for a point on the rigid body $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{a}$. The new inertia tensor is

$$I'_{ij} = \int \rho dV (r'^2 \delta_{ij} - r'_i r'_j) = \int \rho dV \left[ (r^2 + a^2 - 2\boldsymbol{r} \cdot \boldsymbol{a}) \delta_{ij} - (r_i - a_i)(r_j - a_j) \right] = I_{ij} + I^{(a)}_{ij}, \tag{31}$$

$$I^{(a)}_{ij} = m(a^2 \delta_{ij} - a_i a_j). \tag{32}$$

Doing this the other way around, i.e., plug $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{a}$ into the expression of $I_{ij}$ (23), it gives

$$I_{ij} = I'_{ij} + I^{(a)}_{ij} + 2\boldsymbol{D}' \cdot \boldsymbol{a}\delta_{ij} - 2D'_{(i}\, a_{\,j)}, \tag{33}$$

where $\boldsymbol{D}'$ is the mass dipole moment when $O'$ is chosen as the reference point

$$\boldsymbol{D}' = \int \rho(\boldsymbol{r}')\boldsymbol{r}' d^3\boldsymbol{r}'. \tag{34}$$

Comparing (31) to (33) gives a relation satisfied by the mass dipole moment

$$D'_{(i}\, a_{\,j)} - \boldsymbol{D}' \cdot \boldsymbol{a}\, \delta_{ij} = m(a^2\, \delta_{ij} - a_i a_j). \tag{35}$$

Further contracting both sides with $\delta^{ij}$ yields $ma^2 = -\boldsymbol{D}' \cdot \boldsymbol{a}$, or in other words

$$ma = -\boldsymbol{D}' \cdot \hat{\boldsymbol{a}}, \qquad |\boldsymbol{D}'| = ma \equiv D'. \tag{36}$$

The minus sign for the vector expression clearly reflects the geometric setup in Fig 2. Choosing the COM as the reference point, in relativistic language, amounts to choosing the Frenkel-Mathisson-Pirani (FMP) spin suplementary condition (SSC). This can add the correspondence $D^i = 0$ and $D^\mu = 0$ to the correspondence list in Table 1. Although it's not yet clear how (34) can correspond to $D^\mu = -S^{\mu\nu}u_\nu$ in the relativistic case.

The rotational kinetic energy under the new choice is

$$T_{rot} = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega} = \frac{1}{2}\omega^i\omega^j \left[I'_{ij} - m(a^2\delta_{ij} - a_i a_j)\right] = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I}' \cdot \boldsymbol{\omega} - \frac{1}{2}m(\boldsymbol{\omega} \times \boldsymbol{a})^2. \tag{37}$$

The physical intepretation is that, since the new point $O'$ is now considered as a fixed point of rotation, "its kinetic energy" should be subtracted from the rotational kinetic energy and added to the translational kinetic energy. But is the translational kinetic energy increased by the same amount? From (14) we can compute

$$T_{trans} = \frac{1}{2}m\boldsymbol{V}^2 = \frac{1}{2}m(\boldsymbol{V}' - \boldsymbol{\omega} \times \boldsymbol{a})^2 = \frac{1}{2}m\boldsymbol{V}'^2 + \frac{1}{2}m(\boldsymbol{\omega} \times \boldsymbol{a})^2 - m\boldsymbol{V}' \cdot (\boldsymbol{\omega} \times \boldsymbol{a}). \tag{38}$$

The third term, in general, does not vanish. In the end the kinetic energy has to be written as

$$T = \frac{1}{2}m\boldsymbol{V}'^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I}' \cdot \boldsymbol{\omega} + \boldsymbol{D}' \cdot (\boldsymbol{V}' \times \boldsymbol{\omega}), \tag{39}$$

where we have reversed the sign of $\boldsymbol{a}$ in (38), which corresponds to the picture of a net matter distribution direction $\boldsymbol{a}$ measured *from* the new reference point (see Fig 2), and made use of (36). We see that the effect of not choosing COM is to introduce a translation-rotation coupling term in the kinetic energy. This is perhaps why Paul told me only the FMP SSC is the one that is most natural and the one that makes the calculations have more realistic physical meaning. We see that if $\boldsymbol{D} \neq 0$, the conjugate momenta of $\boldsymbol{V}$ and $\boldsymbol{\omega}$, i.e., the linear and angular momenta, will contain the mass dipole moment as an extra dependence on the interior matter distribution.

# 3 Dynamics

Now that the mathematical description of the kinematics is established, it's time to derive the evolution equation that determines the system's dynamics. In the mean time, the rest of the correspondence of Table 1 should be established. Without explicit statement, the reference point is chosen as COM from now on.

Start from (26), the conjugate momenta

$$P_i = \frac{\partial L}{\partial \dot{R}^i} = m\dot{R}_i, \qquad S_i = \frac{\partial L}{\partial \omega^i} = I_{ij}\omega^j \tag{40}$$

are the linear momentum of the COM and (spin) angular momentum of the rigid body itself. The equations of motion are formally written as

$$\dot{P}_i = \frac{\partial L}{\partial R^i} =: F_i, \qquad \dot{S}_i = \frac{\partial L}{\partial \varphi^i} =: N_i. \tag{41}$$

$N_i$ is defined as the torque 1-form exerted on the system. For a specific mechanical system, it's usually given or can be derived. And it being zero means that the system possesses rotational invariance, as is seen by $\partial L/\partial\varphi^i = 0$. To see this geometrically (see Fig 5 of Landau&Lifshitz [1], the following derivation is adapted from it too), for a generic mechanical system (not necessarily a rigid body), imagine an infinitesimal rotation is performed on the system, which can either be a *gedanken* experiment or an actual rotation, the variation of the position vector for a point $A$ is $\delta\boldsymbol{R}_A = \boldsymbol{\delta\varphi} \times \boldsymbol{R}_A$. The variation of the Lagrangian is

$$\begin{aligned}\delta L &= \sum_A \frac{\partial L}{\partial \boldsymbol{R}_A} \cdot \delta\boldsymbol{R}_A + \frac{\partial L}{\partial \dot{\boldsymbol{R}}_A}\delta\dot{\boldsymbol{R}}_A = \sum_A \dot{\boldsymbol{p}}_A \cdot \delta\boldsymbol{R}_A + \boldsymbol{p}_A \cdot \delta\dot{\boldsymbol{R}}_A \\ &= \sum_A \dot{\boldsymbol{p}}_A \cdot (\boldsymbol{\delta\varphi} \times \boldsymbol{R}_A) + \boldsymbol{p}_A \cdot (\boldsymbol{\delta\varphi} \times \dot{\boldsymbol{R}}_A) = \boldsymbol{\varphi} \cdot \sum_A (\boldsymbol{R}_A \times \dot{\boldsymbol{p}}_A) + (\dot{\boldsymbol{R}}_A \times \boldsymbol{p}_A) \\ &= \boldsymbol{\delta\varphi} \cdot \frac{d}{dt}\sum_A (\boldsymbol{R}_A \times \boldsymbol{p}_A),\end{aligned} \tag{42}$$

where the Euler-Lagrangian equation is used. This gives the microscopic interpretation of the angular momentum vector: it's simply $\boldsymbol{R} \times \boldsymbol{p}$ summed over all the points in the system. If the potential is independent of the velocities, the conjugate momentum defined from the Lagrangian comes from the kinetic energy solely, which allows $\dot{\boldsymbol{R}}_A$ to be proportional to $\boldsymbol{p}_A$ with $m_A^{-1}$ being the proportionality constant. This gives a concrete way to calculate the torque

$$\frac{\partial L}{\partial \boldsymbol{\varphi}} = \sum_A \boldsymbol{R}_A \times \boldsymbol{f}_A. \tag{43}$$

It also gives a microscopic interpretation of the torque, although very often we don't have such detailed microscopic information in relativistic mechanics.

For the spin equation of motion, it's also desirable to define its dual 2-form version. A naive guess for the relation between angular momentum and angular velocity 2-forms will be

$$S_{jk} = S^i\epsilon_{ijk} = \frac{\partial L}{\partial \omega^{jk}}. \tag{44}$$

But actually there is a factor of 2 missing. We take (40) and the first equality of (44) as definitions and check the second equality of (44). Since the potential is assumed to be independent of any velocity variable, it suffices to look at the kinetic energy. Rewrite the rotational kinetic energy in angular velocity 2-forms

$$T_{rot} = \frac{1}{2}\omega^i\omega^j I_{ij} = \frac{1}{2}I_{ij}\left(\frac{1}{2}\epsilon^{ikl}\omega_{kl}\right)\left(\frac{1}{2}\epsilon^{jmn}\omega_{mn}\right) = \frac{1}{2}\left(\frac{1}{2}\right)^2 I_{ij}\epsilon^{ikl}\epsilon^{jmn}\omega_{kl}\omega_{mn}. \tag{45}$$

For convenience, we work in a coordinate where for the given time $t$ the inertial tensor $I_{ij}$ is diagonalized, i.e., $I_{ij} = I_i\delta_{ij}$ (no sum over $i$). Use the identity (19), we can establish that

$$T_{rot} = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{1}{2}\right)^2 I_i\left(\delta^{km}\delta^{ln} - \delta^{lm}\delta^{kn}\right)\omega_{kl}\omega_{mn} = \sum_{i=1}^{3}\left(\frac{1}{2}\right)^2 I_i\,\omega_{jk}\omega^{jk}. \tag{46}$$

Then the the partial derivative of the Lagrangian with respect to $\omega^{jk}$ is

$$\frac{\partial L}{\partial \omega^{jk}} = \frac{1}{2}\sum_{i=1}^{3} I_i\,\omega_{jk} = \frac{1}{2}\operatorname{tr}(\boldsymbol{I})\omega_{jk}. \tag{47}$$

On the other hand, the dual of angular momentum $S_i$ is

$$S_{jk} = S^i\epsilon_{ijk} = I^{il}\omega_l\epsilon_{ijk} = \sum_{i=1}^{3} I^i\delta^{il}\omega_l\epsilon_{ijk} = \sum_{i=1}^{3} I^i\omega^l\epsilon_{ljk} = \operatorname{tr}(\boldsymbol{I})\omega_{jk}. \tag{48}$$

Eventually

$$S_{jk} = 2\frac{\partial L}{\partial \omega^{jk}}. \tag{49}$$

Indeed there was a factor of 2 missing in the original guess. The equation of motion for $S_{jk}$ is

$$\dot{S}_{jk} = N^i \epsilon_{ijk}. \tag{50}$$

It's just the Hodge dual of the original equation of motion for $S_i$. Up to now we have established the correspondence of the last three rows of Table 1. Another lesson from this comparison, apart from the one about the mass dipole moment and SSC, is that the spin angular momentum already encodes the information about the matter distribution through the dependence on the inertia tensor. In relativity as we study deformable objects, the internal matter distribution is also hidden in the conjugate momenta $p_\mu$ and $S_{\mu\nu}$ from the general form of the Lagrangian[5]. That's why there is no explicit equation of motion for multipoles higher than dipole. As they captures the internal matter distribution, which is in turn already hidden in $p_\mu$, $S_{\mu\nu}$, the evolution of $p_\mu$, $S_{\mu\nu}$ determines the evolution of the internal matter distribution. However, to close the differential system, we'll need the analog of equation of state for the internal matter distribution. These points should also be present when we study the Newtonian gravity of deformable bodies, which are deformed by spins and tidal fields in the most general case.

The next task is to study two spinning rigid bodies in bound orbits under Newtonian gravity. Can we properly describe its motion and its dynamics? Can we define the total energy and total angular momentum and relate them to the local information of the individual spinning rigid bodies? Can we come up with a variational identities for this relation?

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